RESISTANCE TO LAMINAR FLOW THROUGH POROUS MEDIA[a]

Discussion

John C. Ward,[11] A. M. ASCE. --Rumer and Drinker have bridged the gap between packed beds and single spheres. They have also indicated how fluidized beds behave in this gap. The following development is an attempt to extend their results to higher Reynolds numbers.

An equation applicable for all liquid flow regimes in porous media is[9]

$$\frac{dp}{ds} = \frac{\mu U_s}{k} + \frac{c \rho U_s^2}{\sqrt{k}} \qquad (25)$$

where dp/ds is the pressure drop per unit length due to friction, and c has a value of 0.550 ± 0.024 for unconsolidated porous media. The permeability is given by the Kozeny-Carman equation[12]

$$k = \frac{n^3}{KTa_v} \qquad (26)$$

---

[a] September, 1966, by Ralph R. Rumer, Jr., and Philip A. Drinker (Proc. Paper 4914).

[11] Assoc. Prof. of Civil Engrg., Dept. of Civil Engrg., Colorado State Univ., Fort Collins, Colo.

[12] Corey, A. T., Fluid Mechanics of Porous Solids, Colorado State University, Fort Collins, Colorado, 1965, p. 49.

where K is a dimensionless constant that depends on the shape of the cross section of flow, T is the tortuosity, and a_v is the surface area of dry packing per unit of packed volume. K is exactly 3 for a cross section formed by closely spaced parallel plates and is exactly 2 for a circular cross section. The tortuosity of fully saturated unconsolidated porous media that are isotropic is about 2.

For unconsolidated porous media, Eq. 26 can be written in the following form[13]:

$$k = \frac{n^3 \phi_s^2 M_g^2}{36KT(1-n)^2 \sigma_g^{\ln \sigma_g}} \tag{27}$$

where ϕ_s is the particle shape factor, M_g is the geometric mean particle size, 36 is a pure number, σ_g is the geometric standard deviation of the particle size distribution, and K has a value of 2.36 ± 0.11 for unconsolidated porous media.

An empirical relationship for ϕ_s is

$$\phi_s = \frac{1}{n}\left(\frac{0.198}{\sigma_g} + 0.294\right) - 0.330 \tag{28}$$

which is valid for $n \leqq 0.78$ and $1 \leqq \sigma_g \leqq 2$.

For a bed of uniform diameter spheres, $\sigma_g = 1$, $n = 0.37$, $M_g = d$, and Eq. 27 can be simplified to

---

[13] Closure to reference 9, Volume 92, No. HY4, Proc. Paper 4859, July, 1966, pages 110 - 121.

$$k = \frac{n^3 d^2}{36\ KT\ (1-n)^2} \tag{29}$$

Substituting Eq. 29 into Eq. 25 gives

$$\frac{dp}{ds} = \frac{36KT\ (1-n)^2\ \mu\ U_s}{n^3\ d^2} + \frac{6\sqrt{KT}\ (1-n)\ c\ \rho\ U_s^2}{n^{3/2}\ d} \tag{30}$$

Eq. 14 can be written as follows

$$f_p = C_D \frac{\pi}{8} d^2\ \rho\ u_s^2 \tag{31}$$

The relationship between the actual average velocity of flow within the pore system and U_s is

$$u_s = \frac{U_s \sqrt{T}}{n} \tag{32}$$

because the $\sqrt{T}$ is the ratio of the length of the tortuous path taken by fluid elements to ds.

Eq. 6 can be written as

$$N = \frac{6(1-n)\ d\,A\,d\,s}{\pi\ d^3} \tag{33}$$

and if dz is zero, Eq. 2 can be written

$$\frac{dp}{ds} = \frac{F_R}{n\,dA\;ds} \tag{34}$$

Therefore, combining Eqs. 5, 31, 32, 33, and 34, one obtains

$$\frac{dp}{ds} = \frac{3}{4}\;\frac{C_D\,T\,(1-n)\;\rho\,U_s^2}{n^3\;d} \tag{35}$$

which is the same as Eq. 15 except that C_D in Eq. 35 is equal to the drag coefficient given by Eq. 15 divided by T . Setting Eq. 35 equal to Eq. 30, and solving for C_D , one obtains

$$C_D = \frac{48K\,(1-n)}{R} + 8\;cn^{3/2}\sqrt{\frac{K}{T}} \tag{36}$$

Substituting into Eq. 36 the known values of K , n , c , and T , one obtains

$$C_D = \frac{71.4}{R} + 1.08 \tag{37}$$

For $R \leq 10^4$, an empirical relation for one sphere is [14]

$$C_D = \frac{24}{R} + \frac{3}{\sqrt{R}} + 0.34 \tag{38}$$

---

[14] Fair, G. M. and Geyer, J. C., Water Supply and Waste-Water Disposal. New York: John Wiley and Sons, Inc., 1954, page 586.

Using the data given in Fig. 2, the terminal settling velocity appears to be[15] roughly 15.3 cm/sec corresponding to a Reynolds number of 372. According to Eq. 38, the drag coefficient would have a value of 0.560. This point is plotted in Fig. 3 along with Eq. 38.

The porosity of the expanded bed is given by the following empirical equation[16]

$$n_e = \left(\frac{U_s}{u_t}\right)^{0.22} \qquad (39)$$

where n_e is the porosity of the expanded bed and u_t is the terminal free settling velocity of the spheres. Setting $n_e = 0.37$, the value of the superficial velocity for which the bed begins to expand is roughly 0.167 cm/sec corresponding to a Reynolds number of 4.06. According to Eq. 37, the drag coefficient would have a value of 18.7. This point is plotted in Fig. 3 along with Eq. 37. It should be remembered, as previously noted, that the values of C_D given be Eq. 37 are approximately 1/2 of those given in Fig. 2 for a packed bed.

---

[15] Reference 14, page 588.

[16] Rich, L. G., Unit Operations of Sanitary Engineering. New York: John Wiley and Sons, Inc., 1961, page 149.

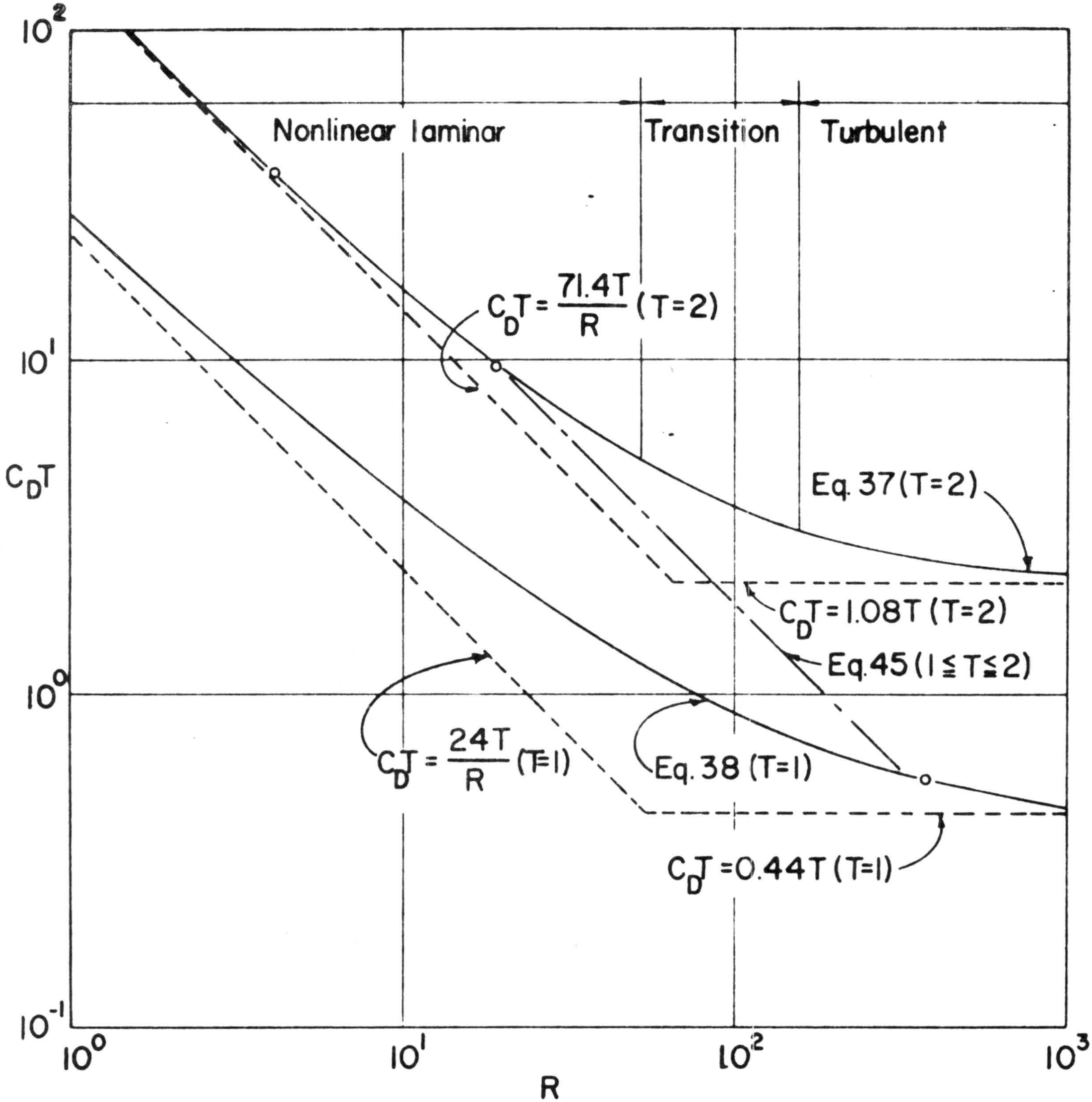

FIG. 3- C_DT Versus R for uniform diameter spheres

At the velocity where a packed bed begins to expand, U_{sc}, the pressure drop, Δp, is given by rewriting Eq. 22 in the following form

$$\Delta p = \rho g \, \Delta s \, (1 - n)(S_s - 1) \tag{40}$$

Further increases in U_s result only in slight increases in Δp. Therefore rewriting Eq. 35 in the form

$$\Delta p = \frac{3}{4} \; \frac{\Delta s \, C_D \, T \, (1 - n) \, \rho \, U_s^2}{n^3 d} \tag{41}$$

and setting it equal to Eq. 40, the result is

$$C_D = \frac{4}{3} \, (S_s - 1) \, \frac{gdn^3}{TU_s^2} \tag{42}$$

which is the same as Eq. 23 except that the value of C_D given by Eq. 42 is equal to the drag coefficient given by Eq. 23 divided by T, as previously noted. If one substitutes into Eq. 42 the known values of $n = 1$, $T = 1$, and $U_s = 15.3$ cm/sec for the fully expanded bed, the value of C_D given by Eq. 42 is 0.498 which is reasonably close to the value given by Eq. 38 which was 0.560.

Setting Eq. 42 equal to Eq. 37, and solving for U_s, one obtains 0.79 cm/sec corresponding to a Reynolds number of 19.2. Both

Eqs. 42 and 37 give a value for C_D of 4.80. This point is also plotted in Fig. 3, and comparison with the experimental data in Fig. 2 indicates that the exponent of 0.22 in Eq. 39 appears to be too low to apply to uniform diameter spheres. A better value appears to be roughly 1/3, so that perhaps Eq. 39 can be written as

$$n_e = \left(\frac{U_s}{u_t}\right)^{1/3} \tag{43}$$

substituting Eq. 43 into Eq. 42 gives

$$C_D T = \frac{4}{3} \frac{(S_s - 1)}{u_t \mu} \frac{g d^2 \rho}{R} \tag{44}$$

Substituting the known values of S_s , d , ρ , u_t , and μ into Eq. 44, one obtains

$$C_D T = \frac{185}{R} \tag{45}$$

Eq. 45 is also plotted in Fig. 3. It should be noted that the value given above for u_t which was 15.3 cm/sec could have easily been either 14 or 16 cm/sec or could be even more inaccurate because of the graphical method used for the determination of u_t . Therefore all values based on u_t would contain the same inaccuracy and may account for some of the minor discrepancies previously observed and the fact that Eq. 45 does not appear to intersect Eq. 38 at quite the correct location. In addition, it would appear that while the line given by Eq. 45 appears to have approximately the correct slope, it probably should be shifted to the right slightly in order to better fit the experimental data.

The weakest link in the development of Eq. 44 is probably Eq. 43. If the authors have values of n_e as a function of Us , it would be very helpful if they could show a plot of log n_e versus $\log\left(\frac{U_s}{u_t}\right)$ using the correct value of u_t to determine the correct value of the exponent in Eq. 43 for uniform diameter spheres. Because Eq. 44 is definitely correct for $U_s = u_t$, any other equation replacing Eq. 44 must be the same as Eq. 44 when $n_e = 1$.

According to McCabe and Smith[17], the value of C_D for one sphere approaches a minimum of 0.44 for $500 \leqq R \leqq 200,000$. This is indicated in Fig. 3.

The relationship between R and R_k can be developed for uniform diameter spheres using Eq. 29

$$R_k = \frac{n^{3/2} \quad R}{6\sqrt{KT} \ (1 - n)} \tag{46}$$

---

[17] McCabe, W. L., and J. C. Smith, Unit Operations of Chemical Engineering, McGraw - Hill Book Co., New York (1956), 360.

and using the known values of n , K , and T gives

$$R_k = 0.0274\ R \tag{47}$$

Flow in porous media can be characterized by four flow regimes [13] which are given in Table 1.

TABLE 1. --POROUS MEDIA FLOW REGIMES

Flow regime	R_k	R for uniform diameter spheres
Linear laminar		
	0.0182	0.665
Nonlinear laminar		
	1.42 ± 0.08	51.9 ± 2.9
Transition		
	4.23 ± 0.51	154 ± 19
Turbulent		

Darcy's law, Eq. 10, is valid only for the linear laminar flow regime. The value of R_k separating the linear laminar and nonlinear laminar flow regimes was arbitrarily taken as the value of R_k for which the quantity dp/ds , as calculated by Eq. 10, is 1% low.

The beginning of the transition flow regime was taken as the value of R_k obtained when turbulence was first observed in any of the visible pore spaces. The beginning of the turbulent flow regime was taken as the value of R_k obtained when all visible pore spaces were just completely turbulent. The values of R_k are the same regardless of whether they are approached by either increasing or decreasing velocity.

The ratio of the Reynolds numbers at the end and beginning of the transition flow regime appears to be somewhere between 2.48 and 3.54 which compares favorably with equivalent ratios of 2 in circular pipes and 2.75 in open conduits. Schneebeli[5] found from visual observations of flow in porous media that the first appearance of turbulence occurred at a value of R of about 60 which compares favorably with the value given in Table 1 when variations in n, ϕ_s, and σ_g are taken into account. Three of the four flow regimes given in Table 1 are shown in Fig. 3 and obviously apply only to Eq. 37.

This writer is somewhat puzzled by the comments of the authors with regard to Eq. 19. First, the range of validity is for $R < 2$. Second, there are no discrepancies between Eqs. 19 and 36 if

$$\xi = 48\,K\,(1 - n) \tag{48}$$

$$C_1 = \frac{c\,n^{3/2}}{6(1-n)\sqrt{KT}} \tag{49}$$

and C_2, C_3, C_4, ... are all zero. Third, the authors have not submitted any experimental data to confirm the validity of Eq. 19. Fourth, Eq. 19 is an approximate solution for the drag coefficient for laminar flow past a sphere and does not necessarily have any relevance to the drag coefficient for porous media. Fifth, Oseen's [18] expression for the drag coefficient for laminar flow past a sphere is

$$C_D = \frac{24}{R} + \frac{9}{2} \qquad (50)$$

which Rumer [18] states, has been shown to be applicable for $R \lessapprox 5$. Therefore Eq. 19 is actually

$$C_D = \frac{\xi}{R} + \xi C_1 \qquad (51)$$

and C_2, C_3, C_4, ... are all zero. Hence, only the first term in the series of Eq. 20 should be retained, and

$$f_k = \frac{1}{R_k} + \frac{d}{\sqrt{k}} C_1 \qquad (52)$$

so that Eq. 21 cannot be obtained from Eq. 20 and therefore Eq. 21 is incorrect and Eq. 52 is the correct expression that should have been written for Eq. 21. By, comparing Eqs. 29 and 49, it is obvious that Eq. 52 becomes

---

[18] Rumer, R. R. Discussion of "Laminar and Turbulent Flow of Water Through Sand," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 90, No. SM2, Proc. Paper 3850, March, 1964, page 205.

$$f_k = \frac{1}{R_k} + c \tag{53}$$

and there is no reason why c should not have a constant value for unconsolidated porous media. It is apparent from Eq. 49 that C_1 is a function of n. There is no experimental evidence to suggest that Eq. 52 is only a partial representation for f_k , and therefore higher order terms are not necessary.

A comparison of Eqs. 13 and 30 shows that

$$C_p = \frac{1}{36\ KT} \tag{54}$$

and therefore C_p is not a function of porosity and does not include the ratio of pore volume to area of solids.

Eq. 31 can be written as follows

$$f_p = \frac{C_D}{8}\ a\ \rho\ u_s^2 \tag{55}$$

where a is the surface area of a sphere. In a circular pipe,

$$f_p = h\ \rho\ g\ \frac{\pi}{4}\ D^2 \tag{56}$$

where D is the diameter of the pipe. The equivalent surface area is

$$a = \pi D s \tag{57}$$

Substitution of Eqs. 56 and 57 into Eq. 55 gives

$$h = C_D \frac{s \ u_s^2}{D \ 2g} \tag{58}$$

and comparison with the Darcy-Weisback equation [19]

$$h = f \frac{s \ u_s^2}{D \ 2g} \tag{59}$$

would indicate that the drag coefficient for circular pipes is the dimensionless friction factor f .

From Fig. 3 it is apparant that the drag coefficient for fluidized beds is still unknown because the variation in the value of T is unknown. However, it should be possible to derive a relationship for the variation of T which could then be verified experimentally. Nevertheless, Rumer and Drinker have opened up a vast new area of research.

---

[19] Reference 14, page 303.