$$\rho(x, t) = \max_{\sigma \in \Sigma} \|e(x, \sigma, t)\|.$$

Comments

a) A symmetrical control structure for $p(x, t)$ is implied by (10). This more restrictive constraint, in comparing with [7] and [8], arises since only some functional properties on $\phi(\cdot)$ are assumed and utilized. A particular example of (10) is

$$p(x, t) = \frac{-\mu(x, t)}{\epsilon} \psi(\rho(x, t)). \tag{11}$$

b) The condition on the uncertainty as shown in (3) is sometimes referred to as a matching condition [8]. Discussions on mismatched uncertainty are in [4], [13], [14].

c) The practicality of (A.3) is preempted by the matching condition. This is so, since one can always *choose* an asymptotically stable nominal part $f_1(x, t)$ and then assume

$$f(x, t) - f_1(x, t) \in \mathcal{R}(B(x, t)) \tag{12}$$

where $\mathcal{R}(B(x, t))$ denotes the range space of $B(x, t)$.

d) As an example, if $\gamma(\|u\|) = b\|u\|^q$, $b > 0$, $q > 1$, then $\psi(\rho) = (\rho/b)^{1/(q-1)}$.

Proof of Theorem: As a consequence of the Caratheodory assumptions on the functions on the right-hand side of (3), one can readily show, using elementary results from the theories of continuous and measurable functions, that

$$g(x, \sigma, t) \triangleq f(x, t) + B(x, t)\phi(p(x, t), \sigma, t) + B(x, t)e(x, \sigma, t) \tag{13}$$

is Caratheodory. Hence, the global existence property is met [10], [11].

For a given $\sigma(\cdot)$, the Lyapunov derivative $\mathcal{L}(\cdot)$: $R^n \times R \to R$ for the closed loop system is given by

$$\mathcal{L}(x, t) = \frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t) f(x, t) + \frac{\partial V}{\partial x}(x, t) B(x, t)[\phi(p(x, t), \sigma(t), t) + e(x, \sigma(t), t)]. \tag{14}$$

Before proceeding, we note that, if $\|\mu\| > \epsilon$, then $u = -\eta\alpha$ where $\eta = \psi(\rho)\rho/\|\mu\|$. Also, utilizing (A.1), we have

$$\alpha^T\phi(-\eta\alpha, \sigma, t) \le \left(\frac{-1}{\eta}\right)\gamma(\psi(\rho)). \tag{15}$$

Furthermore, $\psi(\rho)\rho > 0$ whenever $\|\mu\| > \epsilon$. We also note that $\alpha \ne 0$, since $\|\mu\| > \epsilon$, and $u \ne 0$, since $\|u\| = \psi(\rho) > 0$. If $\|\mu\| \le \epsilon$, suppose $\alpha \ne 0$ and $u \ne 0$, and consider u and α such that $u\|\alpha\| = -\|u\|\alpha$. Then $u = -(\|u\|/\|\alpha\|)\alpha$ and

$$\alpha^T\phi(u, \sigma, t) \le \left(\frac{-\|\alpha\|}{\|u\|}\right)\gamma(\|u\|). \tag{16}$$

Clearly, if $u = 0$, $(\partial V/\partial x)B\phi = 0$ since $\phi(0, \sigma, t) = 0$, and if $\alpha = 0$, $(\partial V/\partial x)B\phi \equiv (\partial V/\partial x)Be = 0$. Thus, if $\|\mu\| > \epsilon$, by (A.3),

$$\mathcal{L}(x, t) \le -\gamma_3(\|x\|) - \frac{\|\mu\|}{\psi(\rho)\rho}\gamma(\psi(\rho)) + \|\mu\| \le -\gamma_3(\|x\|) \tag{17}$$

and if $\|\mu\| \le \epsilon$, $u \ne 0$, $\alpha \ne 0$,

$$\mathcal{L}(x, t) \le -\gamma_3(\|x\|) - \frac{\|\alpha\|}{\|u\|}\gamma(\|u\|) + \|\mu\| \le -\gamma_3(\|x\|) + \epsilon. \tag{18}$$

Also, if $u = 0$ (this implies $\|\mu\| \le \epsilon$),

$$\mathcal{L}(x, t) \le -\gamma_3(\|x\|) + \|\mu\| \le -\gamma_3(\|x\|) + \epsilon \tag{19}$$

and if $\alpha = 0$,

$$\mathcal{L}(x, t) \le -\gamma_3(\|x\|). \tag{20}$$

Consequently, for all $(x, t) \in R^n \times R$,

$$\mathcal{L}(x, t) \le -\gamma_3(\|x\|) + \epsilon. \tag{21}$$

Practical stability then follows [7], [8] by selecting $\mathcal{B}$ to be the closed ball, centered at $x = 0$, with radius $\underline{S} = (\gamma_1^{-1} \circ \gamma_2)(S)$, $S \triangleq \gamma_3^{-1}(\epsilon)$, $\mathcal{C}$ to be the closed ball, centered at $x = 0$, with radius r, and

$$d(\mathcal{C}) = \begin{cases} (\gamma_1^{-1} \circ \gamma_2)(S) & \text{if } r \le S \\ (\gamma_1^{-1} \circ \gamma_2)(r) & \text{if } r > S. \end{cases} \tag{22}$$

Moreover, if $\mathcal{B}$ is the closed ball around $x = 0$ with radius $\bar{S} > (\gamma_1^{-1} \circ \gamma_2)(S)$, then

$$\mathcal{T}(\mathcal{C}, \mathcal{B}) = \begin{cases} 0 & \text{if } r \le (\gamma_2^{-1} \circ \gamma_1)(\bar{S}) \\ \dfrac{\gamma_2(r) - (\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1)(\bar{S})}{(\gamma_3 \circ \gamma_2^{-1} \circ \gamma_1)(\bar{S}) - \epsilon} & \text{otherwise} \end{cases} \tag{23}$$

and $\mathcal{B}_0$ is the closed ball around $x = 0$ with radius $(\gamma_2^{-1} \circ \gamma_1)(\bar{S})$. □

IV. CONCLUSION

This paper shows that the assumption that the uncertain nonlinear system should be with linear input for stabilization [7], [8] is not necessary. In actual design, the amplitude of the control may be constrained, and hence, $\phi(\cdot)$ is of saturation type. This will therefore bring an amplitude limit on the uncertainty. Otherwise, only local performance (e.g., local uniform boundedness) can be assured [13].

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An Efficient Algorithm for Computing the Coefficients of the Characteristic Polynomial of 2-D State-Space Models

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Abstract—**For 2-D systems described by a state-space structure, two different methods for computing the coefficients of their characteristic**

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polynomial are introduced. The relative efficiencies of these methods are compared to that of Faddeeva's method. An efficient method is then chosen which can be used to compute the coefficients of the characteristic polynomial with a minimum number of multiplications.

I. Introduction

In solving theoretical and practical problems associated with systems described by a 2-D state-space structure [1], one often needs to determine the coefficients of the relevant characteristic polynomial. This is particularly important in analyzing the performance of a 2-D system by determining system's sensitivity to the variations of its parameters [2].

In the 1-D case, there are several methods and algorithms available for the temporal case [3] that differ in the amount of computations and the amount of computer storage required. These methods compute the coefficients of the characteristic equation by expanding the secular determinant into a polynomial of degree n.

The 2-D extension of Faddeeva's algorithm [3] is presented by Koo and Chen [4]. This algorithm relates Roesser's 2-D state-space model to the corresponding transfer function of the system. As a result, this method computes both the denominator and the numerator coefficients of the transfer function. In [5] a formula is given which makes use of this algorithm to obtain the transfer function of a 2-D system directly in terms of the state transition matrix and the characteristic polynomial. Although the Faddeeva method serves as a useful tool for translation from state-space structure to transfer function representation, the total number of computations grows largely with the order of the system. In particular, this method becomes very inefficient if only the coefficients of the characteristic polynomial are to be obtained.

Two different methods have been introduced in this paper that provide significant reduction in the amount of computation required. The 2-D extension of Krylov's method [3] is the aim of the first method. This method, which makes use of the 2-D Cayley-Hamilton theorem, reduces the problem of finding the coefficients of the characteristic polynomial to the solution of a system of linear equations that can be performed using standard techniques such as the Gaussian elimination method. In the second method, the idea is to compute the relevant coefficients by finding a sufficiently large number of the numerical values of the secular determinant and then solving a number of standard systems of linear equations. This method offers significant reduction (particularly for large-order systems) in the total number of multiplications, when compared to the Faddeeva and the Krylov extensions. A comparison between the aforementioned methods has been made which reveals the effectiveness of the proposed methods.

II. 2-D System Transfer Function Matrix

Consider a linear time-invariant (LTI) discrete-time 2-D system described by Roesser's state-space model [1], as follows:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i, j) \quad (1a)$$

$$y(i, j) = [C_1 C_2] \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \quad (1b)$$

where $u \in R^1$ and $y \in R^1$ represent the input and output, respectively; $x^h(i, j) \in R^{n_1}$ and $x^v(i, j) \in R^{n_2}$ are the horizontal and the vertical state vectors, respectively; and A_1, A_2, A_3, A_4, B_1, B_2, C_1, and C_2 are constant matrices of appropriate dimensions.

Equation (1) can be expressed more compactly:

$$X'(i, j) = AX(i, j) + Bu(i, j) \quad (2a)$$

$$y(i, j) = CX(i, j) \quad (2b)$$

where the local state $X \in R^n$, $n \triangleq n_1 + n_2$ and

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad C = [C_1 C_2]$$

$$X'(i, j) = \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} \quad \text{and} \quad X(i, j) = \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}.$$

Taking the (z, w)-transform of (1) yields the transfer function

$$H(z, w) = C[S-A]^{-1}B \quad (3)$$

where

$$S = zI_{n_1} \oplus wI_{n_2}$$

and $\oplus$ denotes the direct sum of matrices; I_{n_1} and I_{n_2} are identity matrices of size n_1 and n_2, respectively. To determine $H(z, w)$, let

$$[S-A]^{-1} \triangleq \frac{E(z, w)}{D(z, w)} \quad (4)$$

where

$$D(z, w) = \text{Det } [S-A] = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij} z^{n_1-i} w^{n_2-j} \quad (5)$$

and

$$E(z, w) = \text{Adj } [S-A] = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} F_{ij} z^{n_1-i} w^{n_2-j} \quad (6)$$

where α_{ij}'s are the coefficients of the characteristic polynomial, and F_{ij}'s are $n \times n$ constant matrices. Note that $\alpha_{00} = 1$ and $F_{00} = 0$. The computation of the coefficients α_{ij}'s and matrices F_{ij}'s can be carried out efficiently using the 2-D extension of Faddeeva's method [4]. The total number of multiplications for this algorithm is found to be

$$N_F = (n_1 + n_2)^3 (n_1 + 1)(n_2 + 1). \quad (7)$$

For $n_1 = n_2 = 1, 2, \cdots, 10$, the second column of Table I gives the total number of multiplications, N_F, required to evaluate the coefficients of the characteristic polynomial. This method is particularly useful for translation from a state-space structure to a transfer function representation. However, this technique becomes inefficient, if only the characteristic polynomial should be determined.

In the sequel, two different methods have been introduced which provide more efficient means of computing the coefficients of the characteristic polynomial.

III. 2-D Extension of Krylov's Method

Using the 2-D Cayley-Hamilton theorem [1], the partitioned matrix A satisfies its own characteristic equation, i.e.,

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{ij} A^{(n_1-i),(n_2-j)} = 0 \quad (8)$$

where the state transition matrix $A^{i,j}$ is defined as follows:

1) $A^{-i,j} = A^{i,-j} = 0 \quad$ for $i \geq 1$ and $j \geq 1$

2) $A^{0,0} = I$

3) $A^{i,j} = A^{1,0} A^{i-1,j} + A^{0,1} A^{i,j-1} \quad (9)$

where

$$A^{1,0}=\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A^{0,1}=\begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}.$$

Now, let us take an arbitrary nonzero matrix

$$Y^{0,0}=\begin{bmatrix} y_1^{0,0} & y_2^{0,0} & \cdots & y_n^{0,0} \end{bmatrix}^t$$

where the constituent column vectors $y_i^{0,0}$, $(i = 0, 1, \cdots, n)$ has $[1 + n_1n_2/(n_1 + n_2)]$ elements (the quantity in the square bracket must be rounded up to the next integer above its value). Postmultiplying both sides of (8) by $Y^{0,0}$ yields

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \alpha_{ij} Y^{(n_1-i),(n_2-j)}=0 \tag{10}$$

where

$$Y^{p,q} \triangleq A^{p,q}Y^{0,0}$$
$$\triangleq \begin{bmatrix} y_1^{p,q} & y_2^{p,q} & \cdots & y_n^{p,q} \end{bmatrix}^t$$
$$\text{for } p=0, 1, \cdots, n_1,\ q=0, 1, \cdots, n_2.$$

Using the definition of $A^{i,j}$ in (9), we have

$$Y^{p,q}=A^{1,0}Y^{p-1,q}+A^{0,1}Y^{p,q-1}. \tag{11}$$

As a result, matrices $Y^{p,q}$'s can recursively be computed from (11). Having determined these matrices from (11) and writing (10) in the form of a system of linear equations, we obtain

$$\sum_{\substack{i=0\\ i+j\neq 0}}^{n_1}\sum_{j=0}^{n_2} \alpha_{ij} Z^{(n_1-i),(n_2-j)}= -Z^{n_1,n_2}$$

where

$$Z^{p,q}=[y_1^{p,q\,t} \quad y_2^{p,q\,t} \quad \cdots \quad y_n^{p,q\,t}]^t. \tag{12}$$

Thus, the problem of finding the coefficients $\alpha_{i,j}$'s of the 2-D characteristic polynomial by the Krylov extension reduces to solving the linear system of (12), whose coefficients are computed from (11). Note that the coordinates of the initial vector $Z^{0,0}$ are arbitrarily chosen. Moreover, vectors $Z^{i,j}$ for $(0, 0) \leq (i, j) \leq (n_1, n_2)$, $(i, j) \neq (n_1, n_2)$ form a basis for a complete space, and (12) shows the linear dependency of vector Z^{n_1,n_2} on all the basis vectors. In this case, the minimum polynomial of the matrix coincides with the characteristic polynomial and system (12) has a unique solution and its roots α_{ij} can be computed by standard methods such as the Gaussian elimination technique. If system (12) does not have a unique solution, which occurs because $Z^{i,j}$'s are linearly dependent, it is possible to obtain, in place of the minimum polynomial, some divisors of it [3]. It is obvious that in this case some roots will be lost. The problem of finding the divisors in the 2-D case is complicated [6]. The degeneracy may be avoided by changing the initial vector.

The total number of multiplications required to compute $Y^{p,q}$ matrices is

$$N_1=(n_1+n_2+n_1n_2)(n_1+n_2)^2[1+n_1n_2/(n_1+n_2)]. \tag{13a}$$

If the Gaussian elimination algorithm is used to solve (12), forward and back substitution require, respectively,

$$N_2=(n_1+n_2+n_1n_2)(n_1+n_2+n_1n_2+1)(n_1+n_2+n_1n_2+2)/3$$
$$N_3=(n_1+n_2+n_1n_2)(n_1+n_2+n_1n_2-1)/2 \tag{13b}$$

TABLE I
NUMBER OF MULTIPLICATIONS FOR DIFFERENT METHODS

n_1, n_2	N_F	N_K	N_E
(1,1)	36	47	38
(2,2)	585	524	251
(3,3)	3472	3085	872
(4,4)	12825	10084	2229
(5,5)	36036	30135	4746
(6,6)	84721	67976	8943
(7,7)	175680	151053	15436
(8,8)	331857	282680	24937
(9,9)	583300	530607	38254
(10,10)	968121	885620	56291

multiplications and divisions. Therefore, the total number of multiplications and divisions for this method is

$$N_K=N_1+N_2+N_3. \tag{13c}$$

This is tabulated, for the same values of n_1 and n_2, in the third column of Table I. As can be seen, the 2-D extension of Krylov's method reduces the computational effort by 10 percent, when compared to that of Faddeeva's algorithm.

IV. AN EFFICIENT ALGORITHM

The secular determinant $D(u, v)$ can also be expanded by finding a sufficiently large number of its numerical values. Setting $u = 0$ and $v = 0, 1, \cdots, n_2$ successively, gives

$$\alpha_{n_1n_2}=D(0, 0)$$

and

$$\Gamma_{n_2}\theta_{n_1}=\Delta_0 \tag{14}$$

where

$$\Gamma_{n_2}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^{n_2} & 2^{n_2-1} & \cdots & 2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n_2^{n_2} & n_2^{n_2-1} & \cdots & n_2 \end{bmatrix}, \quad \theta_{n_1}=\begin{bmatrix} \alpha_{n_10} \\ \alpha_{n_11} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n_1n_2-1} \end{bmatrix}$$

and

$$\Delta_0=\begin{bmatrix} D(0, 1)-D(0, 0) \\ D(0, 2)-D(0, 0) \\ \cdot \\ \cdot \\ \cdot \\ D(0, n_2)-D(0, 0) \end{bmatrix}.$$

Similarly for $u = 1$ and $v = 0, 1, \cdots, n_2$, we have

$$\alpha_{0n_2}+\alpha_{1n_2}+\alpha_{2n_2}+\cdots+\alpha_{n_1n_2}=D(1, 0)$$

and

$$\Gamma_{n_2}P^{(1)}=\Delta_1 \tag{15}$$

where

$$P^{(1)}=\begin{bmatrix} p_0^{(1)} \\ p_1^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ p_{n_2-1}^{(1)} \end{bmatrix}, \quad \Delta_1=\begin{bmatrix} D(1, 1)-D(1, 0) \\ D(1, 2)-D(1, 0) \\ \cdot \\ \cdot \\ \cdot \\ D(1, n_2)-D(1, 0) \end{bmatrix}$$

and

$$p_i^{(1)} = \alpha_{0i} + \alpha_{1i} + \cdots + \alpha_{n_1 i}$$

for $i = 0, 1, \cdots, n_2 - 1$. The same procedure can be repeated for u, taking successive integer values up to n_1, i.e.,

$$\alpha_{0n_2} n_1^{n_1} + \alpha_{1n_2} n_1^{n_1 - 1} + \cdots + \alpha_{n_1 n_2} = D(n_1, 0)$$

and

$$\Gamma_{n_2} P^{(n_1)} = \Delta_{n_1} \tag{16}$$

where

$$P^{(n_1)} = \begin{bmatrix} p_0^{(n_1)} \\ p_1^{(n_1)} \\ \cdot \\ \cdot \\ \cdot \\ p_{n_2-1}^{(n_1)} \end{bmatrix}, \quad \Delta_{n_1} = \begin{bmatrix} D(n_1, 1) - D(n_1, 0) \\ D(n_1, 2) - D(n_1, 0) \\ \cdot \\ \cdot \\ \cdot \\ D(n_1, n_2) - D(n_1, 0) \end{bmatrix}$$

and

$$p_i^{(n_1)} = n_1^{n_1} \alpha_{0i} + n_1^{n_1 - 1} \alpha_{1i} + \cdots + \alpha_{n_1 i}.$$

Having found the set of vectors $P^{(1)} \cdots P^{(n_1)}$, the coefficient α_{ij}'s of the characteristic polynomial can then be computed using the following set of systems of linear equations.

$$\Gamma_{n_1} \theta_0' = \Psi_0 \tag{17}$$

where

$$\theta_0' = \begin{bmatrix} 1 \\ \alpha_{10} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{(n_1-1)0} \end{bmatrix}, \quad \Psi_0 = \begin{bmatrix} p_0^{(1)} - \alpha_{n_1 0} \\ p_0^{(2)} - \alpha_{n_1 0} \\ \cdot \\ \cdot \\ \cdot \\ p_0^{(n_1)} - \alpha_{n_1 0} \end{bmatrix}$$

and Γ_{n_1} has a form similar to Γ_{n_2}. Likewise

$$\Gamma_{n_1} \theta_1' = \Psi_1 \tag{18}$$

where

$$\theta_1' = \begin{bmatrix} \alpha_{01} \\ \alpha_{11} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{(n_1-1)1} \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} p_1^{(1)} - \alpha_{n_1 1} \\ p_1^{(2)} - \alpha_{n_1 1} \\ \cdot \\ \cdot \\ \cdot \\ p_1^{(n_1)} - \alpha_{n_1 1} \end{bmatrix}.$$

Repeating the same procedure finally results in

$$\Gamma_{n_1} \theta_{n_2}' = \Psi_{n_2} \tag{19}$$

where

$$\theta_{n_2}' = \begin{bmatrix} \alpha_{0n_2} \\ \alpha_{1n_2} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{(n_1-1)n_2} \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} p_{n_2}^{(1)} - \alpha_{n_1 n_2} \\ p_{n_2}^{(2)} - \alpha_{n_1 n_2} \\ \cdot \\ \cdot \\ \cdot \\ p_{n_2}^{(n_1)} - \alpha_{n_1 n_2} \end{bmatrix}.$$

Note that $\alpha_{n_1 1}$ in Ψ_1 is computed via (14). As a consequence, the coefficient α_{ij} can be computed by first finding $(n_1 + 1)(n_2 + 1)$ numerical values of $D(u, v)$ and then solving systems of linear equations (14)–(19).

Assuming that the Gaussian elimination technique is employed to solve (14)–(16), a significant saving in the number of multiplications may be achieved in the forward substitution step, since matrix Γ_{n_2} is common to all of these systems. In this case, a total of

$$M_1 = n_2(n_2+1)(2n_2+1)/6 + n_2^2(n_1+1) \tag{20a}$$

multiplications and divisions is required. Similarly, to solve systems (17)–(19), a total of

$$M_2 = n_1(n_1+1)(2n_1+1)/6 + n_1^2(n_2+1) \tag{20b}$$

multiplications and divisions is required.

Gaussian elimination can also be used to evaluate the numerical values of $D(u, v)$. This requires a total of

$$M_3 = (n_1+1)(n_2+1)(n_1+n_2)(n_1+n_2+2) \tag{20c}$$

multiplications and divisions. Thus, the total number of multiplications for this method will be

$$N_E = M_1 + M_2 + M_3. \tag{20d}$$

The numerical values of N_E are reflected in the fourth column of Table I. As can be seen, this method offers great reduction (approx. 90 percent) in the total number of multiplications, when compared to Faddeeva's and Krylov's extensions.

V. Conclusions

Two different methods for finding the coefficients of the characteristic polynomial of 2-D discrete time-systems have been introduced. A comparison of the relative effectiveness of these methods is made based on the total number of multiplications required. The second method proposed in this paper is shown to provide a very efficient means of computing the relevant coefficients, for systems with any order.

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A Control Scheme for a Class of Discrete Nonlinear Stochastic Systems

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Abstract—**A control scheme is presented for effective stabilization of discrete-time, nonlinear stochastic systems where the nonlinearity in-**

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