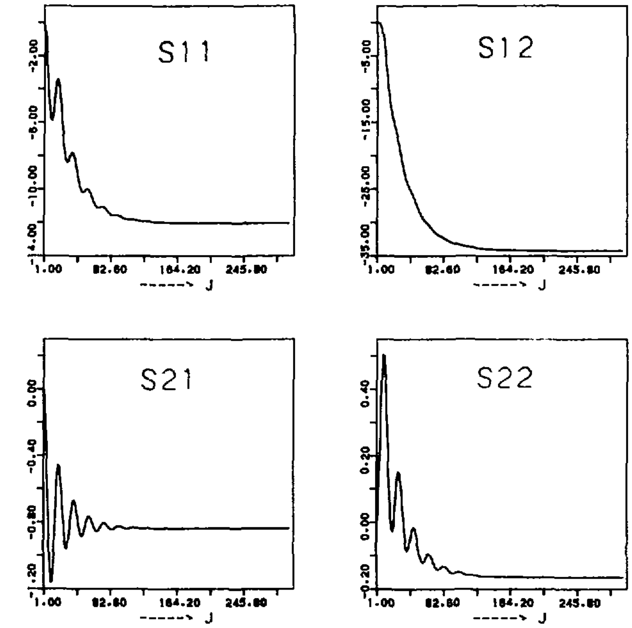

Fig. 1. Open-loop step response for the delta wing aircraft model.

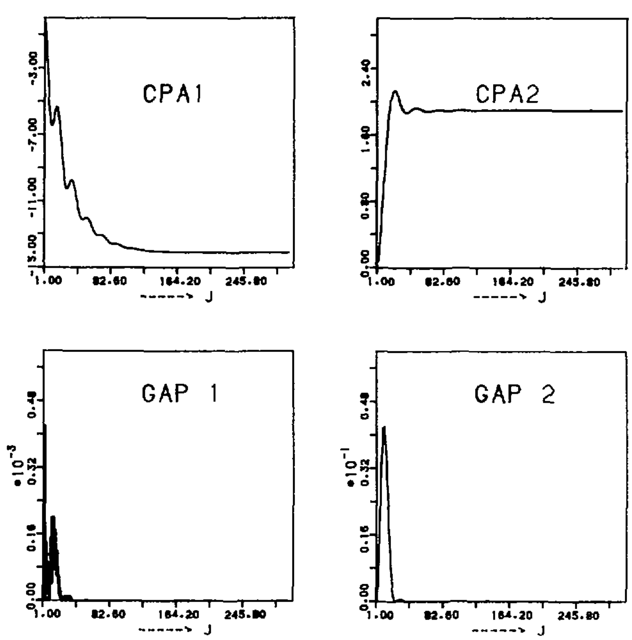

Fig. 2. Open-loop step CPA and CG of the delta wing aircraft model.

CPA and CVE for systems with many inputs and outputs directly from step or impulse data, thereby bypassing the parameter estimation problem. As shown in [4], [5], the displayed CPA and CVE reveal important properties of the time-domain input–output modes which are valuable for multivariable control system analysis and design.

Appendix

To prove that (7) is consistent.

$$Z_i(1)w_i(2) = -Z_i(2)w_i(1)$$

$$[\lambda_i(1)I - G(1)]w_i(2) = -[\lambda_i(2)I - G(2)]w_i(1)$$

where $G(j)$ is an $n \times n$ matrix of rank n. Since $\lambda_i(1)$ is one of the distinct eigenvalues of $G(1)$, the matrix $[\lambda_i(1)I - G(1)]$ is of rank $n - 1$.

By definition of the dual eigenvector:

$$v_i^t(1)[\lambda_i(1)I - G(1)] = 0$$

since $v_i^t(1)$ is orthogonal to the space spanned by $[\lambda_i(1)I - G(1)]$.
We also have

$$v_i^t(1)[\lambda_i(2)I - G(2)]w_i(1) = \lambda(2) - v_i^t(1)G(2)w_i(1) = 0.$$

Therefore, the column vector $[\lambda_i(2)I - G(2)]w_i(1)$ is also orthogonal to $v_i^t(1)$ and, hence, must lie in the $(n - 1)$-dimensional space spanned by $[\lambda_i(1)I - G(1)]$. The same argument can be extended for all the sampling instants.

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Feedback Control of Two-Time-Scale Block Implemented Discrete-Time Systems

K. KHORASANI AND M. R. AZIMI-SADJADI

Abstract—**The block processing technique is applied to a class of two-time-scale linear discrete-time systems. Block state-space structures for slow and fast subsystems have been obtained. Based upon these models, blocked slow and fast controls are designed. In addition, an extra control is designed to exactly compensate for the effects of the parasitics.**

I. Introduction

Block processing was originally introduced by Burrus [1] as an efficient method for implementation of recursive digital filters, especially when used in conjunction with fast transform techniques. He used matrix representation of linear filtering operations and obtained different block recursive structures in the form of state-variable with block feedback. More recently, Barnes and Shinnaka [2] have derived a 1-D multiinput, multioutput (MIMO) block state-space structure from a 1-D single-input, single-output (SISO) state-space model. They have shown that the poles of the block state-space model are the Nth power (N being the block size) of the poles of an associated state-space realization, i.e., if the original SISO system is assumed to be stable, then the poles of the MIMO block implemented filter move closer to the origin of the unit circle as the block length is increased. They have also shown [3] that the block state-space exhibits reduced roundoff error and eigenvalue sensitivity characteristics when compared to those of its scalar counterpart.

In this paper, the block processing technique is applied to a class of

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linear discrete-time systems which exhibit two-time-scale behavior. It is shown that, using this scheme, the slow dynamics will remain slow while the fast dynamics become faster. By a proper choice of the block size N it is possible to design an additional compensating control which can be applied to the full system to compensate exactly for the effects of the parasitics.

II. Singularly Perturbed Discrete-Time Systems

Consider the following linear discrete-time singularly perturbed system which has been treated in [4]:

$$\begin{bmatrix} x(n+1) \\ z(n+1) \end{bmatrix} = \begin{bmatrix} (I_1+\epsilon A_{11}) & \epsilon A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(n) \\ z(n) \end{bmatrix} + \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix} u(n) \quad (1)$$

$$y(n) = C_1 x(n) + C_2 z(n) \quad (2)$$

where $x \in R^p$, $z \in R^m$, $u \in R$, $y \in R$, and $\epsilon > 0$ is a small perturbation parameter. The system in (1) and (2) is said to be expressed in the fast time scale n [4]. Vectors x and z represent the slow and fast states, respectively. The decoupling transformation [5] can be utilized to separate the slow and fast dynamics. Then, we have

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = P \begin{bmatrix} x \\ z \end{bmatrix} \quad (3)$$

where

$$P = \begin{bmatrix} I_1 - \epsilon ML & -\epsilon M \\ L & I_2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} I_1 & \epsilon M \\ -L & I_2 - \epsilon LM \end{bmatrix}. \quad (4)$$

Application of (3) and (4) to (1) and (2) yields

$$\begin{bmatrix} w_1(n+1) \\ w_2(n+1) \end{bmatrix} = \begin{bmatrix} I_1 + \epsilon A_{11} - \epsilon A_{12} L & 0 \\ 0 & A_{22} + \epsilon L A_{12} \end{bmatrix} \begin{bmatrix} w_1(n) \\ w_2(n) \end{bmatrix} + \begin{bmatrix} \epsilon B_1 - \epsilon M B_2 - \epsilon^2 M L B_1 \\ B_2 + \epsilon L B_1 \end{bmatrix} u(n) \quad (5)$$

$$y(n) = [C_1 - C_2 L \quad C_2 + \epsilon C_1 M - \epsilon C_2 L M][w_1'(n) \quad w_2'(n)]' \quad (6)$$

where matrices L (size $m \times p$) and M (size $p \times m$) are the real roots of

$$A_{22}L - L - A_{21} + \epsilon L A_{12} L - \epsilon L A_{11} = 0$$

$$-MA_{22} + M + A_{12} + \epsilon A_{11} M - \epsilon A_{12} L M - \epsilon M L A_{12} = 0$$

respectively. It can easily be shown that

$$L(\epsilon) := L = -(I_2 - A_{22})^{-1} A_{21} + \epsilon L_r(\epsilon)$$

where L_r satisfies

$$-A_{22}L_r + L_r + \epsilon L_r A_{11} - (I_2 - A_{22})^{-1} A_{21} A_s - \epsilon^2 L_r A_{12} L_r + \epsilon (I_2 - A_{22})^{-1} A_{21} A_{12} L_r + \epsilon L_r A_{12} (I_2 - A_{22})^{-1} A_{21} = 0$$

with

$$A_s := A_{11} + A_{12}(I_2 - A_{22})^{-1} A_{21}$$

and

$$M(\epsilon) := M = -A_{12}(I_2 - A_{22})^{-1} + \epsilon M_r(\epsilon)$$

where M_r satisfies

$$M_r - M_r A_{22} - (A_{11} - A_{12}L) A_{12} (I_2 - A_{22})^{-1} + A_{12}(I_2 - A_{22})^{-1} L A_{12} + \epsilon (A_{11} - A_{12} L) M_r - \epsilon M_r L A_{12} = 0.$$

Thus, the decoupled form of (5) and (6) becomes

$$\begin{bmatrix} w_1(n+1) \\ w_2(n+1) \end{bmatrix} = \begin{bmatrix} I_1 + \epsilon A_s - \epsilon^2 A_{12} L_r & 0 \\ 0 & A_{22} - \epsilon (I_2 - A_{22})^{-1} A_{21} A_{12} + \epsilon^2 L_r A_{12} \end{bmatrix} \cdot \begin{bmatrix} w_1(n) \\ w_2(n) \end{bmatrix} + \begin{bmatrix} \epsilon B_s - \epsilon^2 (M L B_1 + M_r B_2) \\ B_2 - \epsilon (I_2 - A_{22})^{-1} A_{21} B_1 + \epsilon^2 L_r B_1 \end{bmatrix} u(n) \quad (7)$$

$$y(n) = [C_1 + C_2(I_2 - A_{22})^{-1} A_{21} - \epsilon C_2 L_r \quad C_2 + \epsilon C_1 M - \epsilon C_2 L M] \begin{bmatrix} w_1(n) \\ w_2(n) \end{bmatrix} \quad (8)$$

where $B_s := B_1 + A_{12}(I_2 - A_{22})^{-1} B_2$.

Subsequently, the slow and fast subsystems for (1) and (2) can be shown [4] to be

$$x_s(n+1) = (I_1 + \epsilon A_s) x_s(n) + \epsilon B_s u_s(n) \quad (9)$$

$$y_s(n) = C_s x_s(n) \quad (10)$$

and

$$z_f(n+1) = A_{22} z_f(n) + B_2 u_f(n) \quad (11)$$

$$y_f(n) = C_2 z_f(n) \quad (12)$$

respectively, where

$$C_s := C_1 + C_2(I_2 - A_{22})^{-1} A_{21}, \quad z_f = z - (I_2 - A_{22})^{-1}(A_{21} x_s + B_2 u_s).$$

The slow and fast controls, $u_s(n)$ and $u_f(n)$, are designed based upon the slow subsystem (9), (10) and the fast subsystem (11), (12) to stabilize (9)–(12).

Assumption 1: The pair (A_s, B_s) is stabilizable in the continuous time sense, i.e., the unstable subspace is contained in its controllable subspace; and the pair (A_{22}, B_2) is stabilizable in the discrete-time sense, i.e., all the characteristic values of the matrix A_{22} that have moduli greater than 1 are controllable.

Lemma 1 [6]: Under Assumption 1, there exists an $\epsilon^* > 0$ such that the full system (1), (2) is stabilizable in the discrete-time sense for all $\epsilon \in (0, \epsilon^*]$.

In the sequel, a MIMO state-space model for the singularly perturbed system is derived utilizing the block processing scheme [2].

III. Block Implementation of Singularly Perturbed Discrete-Time Systems

Assume that the input and output sequences are sectioned into blocks of size N. If the state associated with each block is defined by the state at the edge element of the block, the state-space equations in (7) and (8) can be represented in block form as

$$\begin{bmatrix} W_1(n+1) \\ W_2(n+1) \end{bmatrix} = \begin{bmatrix} A_{11}^N & 0 \\ 0 & A_{22}^N \end{bmatrix} \begin{bmatrix} W_1(n) \\ W_2(n) \end{bmatrix} + \hat{B} U(n) \quad (13)$$

$$Y(n) = \hat{C}[W_1(n)' \quad W_2(n)']' \quad (14)$$

where

$$W_1(n) = w_1(nN), \quad W_2(n) = w_2(nN)$$

$$U(n) = [u(nN) \quad u(nN+1) \cdots u(nN+N-1)]'$$

and similarly for $Y(n)$. Also we have

$$A_{11} := I_1 + \epsilon A_s - \epsilon^2 A_{12} L_r$$

$$A_{22} := A_{22} - \epsilon(I_2 - A_{22})^{-1}A_{21}A_{12} + \epsilon^2 L_r A_{12}$$

$$\hat{B} := \begin{bmatrix} A_{11}^{N-1}B_{10} & A_{11}^{N-2}B_{10} & \cdots & B_{10} \\ A_{22}^{N-1}B_{20} & A_{22}^{N-2}B_{20} & & B_{20} \end{bmatrix}$$

$$B_{10} := \epsilon B_s - \epsilon^2(MLB_1 + M_r B_2)$$

$$B_{20} := B_2 - \epsilon(I_2 - A_{22})^{-1}A_{21}B_1 + \epsilon^2 L_r B_1$$

$$\hat{C} := \begin{bmatrix} C_{10} & C_{20} \\ C_{10}A_{11} & C_{20}A_{22} \\ \vdots & \\ C_{10}A_{11}^{N-1} & C_{20}A_{22}^{N-1} \end{bmatrix}$$

$$C_{10} := C_1 + C_2(I_2 - A_{22})^{-1}A_{21} - \epsilon C_2 L_r$$

$$C_{20} := C_2 + \epsilon C_1 M - \epsilon C_2 LM.$$

Using (13), the state vector is updated only at the beginning of each block while the nth output block is being evaluated. This can be interpreted as a multirate system with reduced sampling rate. In view of Lemma 1 and assuming that $N = p + m$, it is observed that matrix $\hat{B}$ becomes nonsingular.

Now, let us rewrite (13) as

$$\begin{bmatrix} W_1(n+1) \\ W_2(n+1) \end{bmatrix} = \begin{bmatrix} (I_1 + \epsilon A_s)^N + R_s & 0 \\ 0 & A_{22}^N + R_f \end{bmatrix} \begin{bmatrix} W_1(n) \\ W_2(n) \end{bmatrix} + \hat{B}U(n) \tag{15}$$

$$Y(n) = \hat{C}[W_1(n)' \quad W_2(n)']'$$

where

$$R_s := (I_1 + \epsilon A_s - \epsilon^2 A_{12}L_r)^N - (I_1 + \epsilon A_s)^N = 0(\epsilon^2)$$

$$R_f := (A_{22} - \epsilon(I_2 - A_{22})^{-1}A_{21}A_{12} + \epsilon^2 L_r A_{12})^N - A_{22}^N = 0(\epsilon).$$

The block representation of the slow and fast subsystems (9)–(12) are also given by

$$X_s(n+1) = (I_1 + \epsilon A_s)^N X_s(n) + \epsilon \hat{B}_s U_s(n) \tag{16}$$

$$Y_s(n) = \hat{C}_s X_s(n) \tag{17}$$

and

$$Z_f(n+1) = A_{22}^N Z_f(n) + \hat{B}_f U_f(n) \tag{18}$$

$$Y_f(n) = \hat{C}_f Z_f(n) \tag{19}$$

where

$$X_s(n) = x_s(nN), \quad Z_f(n) = z_f(nN)$$

$$Y_s(n) = [y_s(nN) \quad y_s(nN+1) \cdots y_s(nN+N-1)]'$$

$$U_s(n) = [u_s(nN) \quad u_s(nN+1) \cdots u_s(nN+N-1)]'$$

with $Y_f(n)$ and $U_f(n)$ defined in a similar manner. Also we have

$$\hat{B}_s := [(I_1 + \epsilon A_s)^{N-1}B_s \quad (I_1 + \epsilon A_s)^{N-2}B_s \cdots B_s]$$

$$\hat{B}_f := [A_{22}^{N-1}B_2 \quad A_{22}^{N-2}B_2 \cdots B_2]$$

$$\hat{C}_s := [C_s \quad C_s(I_1 + \epsilon A_s) \cdots C_s(I_1 + \epsilon A_s)^{N-1}]$$

$$\hat{C}_f := [C_2 \quad C_2 A_{22} \cdots C_2 A_{22}^{N-1}].$$

In the following section, the problem of designing the blocked slow and fast controls for (16), (17) and (18), (19) is addressed.

IV. Block Feedback Design

Two cases have been considered in this section. In Section IV-A, feedback control laws are designed for separate slow and fast subsystems assuming full state measurements. Then, in Section IV-B, output feedback control laws are obtained when the state measurements are not directly available for feedback.

A. State Feedback

The state feedback law for the slow control $u_s(n)$ is designed as

$$u_s(n) = K_s x_s(n) \tag{20}$$

where K_s is chosen so that Re $\lambda\{A_s + B_s K_s\} < 0$ and for sufficiently small ϵ, (9) is asymptotically stable, that is, $x_s(n) \to 0$ as $n \to \infty$. The elements of the block $U_s(n)$ can now be expressed in terms of the state $x_s(nN)$ associated with the nth block, i.e.,

$$u_s(nN) = K_s x_s(nN)$$

$$u_s(nN+1) = K_s x_s(nN+1) = K_s(I_1 + \epsilon A_s + \epsilon B_s K_s) x_s(nN)$$

$$\vdots$$

$$u_s(nN+N-1) = K_s(I_1 + \epsilon A_s + \epsilon B_s K_s)^{N-1} x_s(nN). \tag{21}$$

Arranging (21) in the vector form, the blocked slow control will become

$$U_s(n) = \mathbf{K}_s X_s(n) \tag{22}$$

where

$$\mathbf{K}_s = [(K_s)' \quad (K_s(I_1 + \epsilon A_s + \epsilon B_s K_s))' \cdots (K_s(I_1 + \epsilon A_s + \epsilon B_s K_s)^{N-1})']'.$$

Now, applying the blocked slow control $U_s(n)$ to (16), the closed loop system becomes

$$X_s(n+1) = (I_1 + \epsilon A_s)^N X_s(n) + \epsilon \sum_{i=0}^{N-1} (I_1 + \epsilon A_s)^{N-1-i} B_s K_s (I + \epsilon A_s + \epsilon B_s K_s)^i X_s(n). \tag{23}$$

Equation (23) can be simplified to

$$X_s(n+1) = (I_1 + \epsilon A_s + \epsilon B_s K_s)^N X_s(n). \tag{24}$$

Let the state feedback law for the fast control $u_f(n)$ be given by

$$u_f(n) = K_f z_f(n) \tag{25}$$

where K_f is chosen such that $|\lambda\{A_{22} + B_2 K_f\}| < 1$, that is, (13) is asymptotically stable so that $z_f(n) \to 0$ as $n \to \infty$. The blocked fast control can be defined similarly to that of the slow control, that is,

$$U_f(n) = \mathbf{K}_f Z_f(n) \tag{26}$$

where

$$\mathbf{K}_f = [(K_f)' \quad (K_f(A_{22} + B_2 K_f))' \cdots (K_f(A_{22} + B_2 K_f)^{N-1})']'.$$

Applying (26) to (18), the resulting closed loop system becomes

$$Z_f(n+1) = (A_{22} + B_2 K_f)^N Z_f(n). \tag{27}$$

Remark 1: A consequence of the above result is that if $\lambda_i(\epsilon) := 1 + \epsilon s_i(\epsilon)$, $i = 1, 2, \cdots, p$ and $\lambda_{j+p}(\epsilon) := f_j(\epsilon)$, $j = 1, 2, \cdots, m$ are the eigenvalues of the closed loop system (1) with state feedback laws (20) and (25), then examining the eigenvalues of the blocked system (24) and (27) reveals that the slow eigenvalues λ_i^N, $i = 1, \cdots, p$ remain close to the unit circle, whereas the fast eigenvalues λ_{j+p}^N, $j = 1, \cdots, m$ move closer to the origin.

The design objectives for the slow and fast subsystems can be met by applying the blocked slow and fast control strategies developed in (22) and (26) to (16)–(19). However, the ultimate goal is to ensure that the design objectives are satisfied for the full system. This can be accomplished by using an extra compensating control. Therefore, $U(n)$ will be

$$U(n) = -\hat{B}^{-1}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{bmatrix} X(n) \\ Z(n) \end{bmatrix} + \hat{B}^{-1}\begin{bmatrix} \epsilon\hat{B}_s K_s X(n) \\ \hat{B}_f K_f Z(n) \end{bmatrix} \quad (28)$$

$$T_1 := R_s + \epsilon M A_{22}^N L - \epsilon A_{11}^N M L$$

$$T_2 := \epsilon M A_{22}^N - \epsilon A_{11}^N M$$

$$T_3 := (I_2 - \epsilon L M) A_{22}^N L - L A_{11}^N (I_1 - \epsilon M L)$$

$$T_4 := R_f - \epsilon L M A_{22}^N + \epsilon L A_{11}^N M$$

where the first term in (28) represents the compensating control and the second term represents the reduced controls (22) and (25) with X_s and Z_f replaced by X and Z, respectively. It can be shown that by applying $U(n)$ to (15), the full closed loop system becomes

$$X(n+1) = (I_1 + \epsilon A_s + \epsilon B_S K_s)^N X(n) \quad (29)$$

$$Z(n+1) = (A_{22} + B_f K_f)^N Z(n).$$

As a consequence, the following result is established.

Theorem 1: Let Lemma 1 hold and assume that $N = p + m$; then there exists an $\epsilon^* > 0$ such that (15), with the control law $U(n)$ given by (28), is asymptotically stable for all $\epsilon \in (0, \epsilon^*]$. Furthermore, the following relations hold uniformly for all $n \geq 0$:

$$X(n) = X_s(n) \quad (30)$$

$$Z(n) = Z_f(n). \quad (31)$$

Comparing the results developed in Theorem 1 to those of the standard methods indicates that the slow and fast dynamics contain purely slow and fast parts, respectively.

Remark 2: An approximate compensating control can be obtained by retaining the $O(\epsilon)$ terms in M and L matrices (consequently in T_i's) and neglecting the higher order terms in ϵ. This approximation, which reduces the amount of required computation for matrices T_i, $i = 1, \cdots, 4$, would yield $X(n) = X_s(n) + O(\epsilon^2)$ and $Z(n) = Z_f(n) + O(\epsilon^2)$ uniformly for all $n \geq 0$.

Remark 3: The exact elimination of the effects of the parasitic should come as no surprise, because it could be derived independently of singular perturbation theory by performing eigenvalue placement on the entire blocked full order system. Since the full order system is completely controllable for sufficiently small ϵ, the eigenvalues of the blocked system can be placed arbitrarily.

B. Output Feedback

If the state measurements are not directly available for feedback, measurements can be taken from the output. The output feedback laws for slow and fast controls are designed as

$$u_s(n) = H_s y_s(n) \quad (32)$$

$$u_f(n) = H_f y_f(n). \quad (33)$$

Assumption 2: The pairs $\{(I + \epsilon A_s), C_s\}$ and $\{A_{22}, C_2\}$ are detectable in the discrete-time sense, i.e., the unobservable subspace is contained in the stable subspace.

The output feedback gains H_s and H_f are chosen such that Re $\lambda\{A_s + B_s H_s C_s\} < 0$, and $|\lambda\{A_{22} + B_2 H_f C_f\}| < 1$, that is, the slow and fast subsystems (9)–(12) are asymptotically stable.

Lemma 2: Under Assumption 2, there exists an $\epsilon^* > 0$ such that the full system (1), (2) is detectable in the discrete-time sense for all $\epsilon \in (0, \epsilon^*]$.

If the block size is chosen to be $N = p + m$, and Lemma 2 holds, then the $N \times N$ matrix $\hat{C}$ is nonsingular. Applying the block processing scheme to (32) and (33) and using (9), the blocked slow and fast controls are designed as follows:

$$U_s(n) = H_s Y_s(n) = H_s \hat{C}_s X_s(n) \quad (34)$$

where

$$H_s\hat{C}_s := [(H_s C_s)' \quad (H_s C_s(I + \epsilon A_s + \epsilon B_s H_s C_s))' \cdots (H_s C_s \cdot (I + \epsilon A_s + \epsilon B_s H_s C_s)^{N-1})']'$$

and

$$U_f(n) = H_f Y_f(n) = H_f \hat{C}_f Z_f(n) \quad (35)$$

where

$$H_f\hat{C}_f := [(H_f C_f)' \quad (H_f C_f(A_{22} + B_2 H_f C_f))' \cdots (H_f C_f \cdot (A_{22} + B_2 H_f C_f)^{N-1})']'.$$

Applying the blocked slow and fast controls (34) and (35) to (16)–(19), the resulting closed loop system becomes

$$X_s(n+1) = (I + \epsilon A_s + \epsilon B_s H_s C_s)^N X_s(n) \quad (36)$$

$$Z_f(n+1) = (A_{22} + B_2 H_f C_f)^N Z_f(n). \quad (37)$$

To ensure that the full design objectives are satisfied for the full system (15), the overall control $U(n)$ is chosen to be

$$U(n) = -\hat{B}^{-1}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \hat{P}^{-1}\hat{C}^{-1} Y(n) + \hat{B}^{-1}\begin{bmatrix} \epsilon\hat{B}_s H_s \hat{C}_s & 0 \\ 0 & \hat{B}_f H_f \hat{C}_f \end{bmatrix} P^{-1}\hat{C}^{-1} Y(n) \quad (38)$$

where the first term is the compensating control with T_1, T_2, T_3, and T_4 as defined in (28), and the second term is the reduced slow and fast controls (34) and (35) obtained by first replacing X_s and Z_f by X and Z, respectively, and then using $[X(n)' \ Z(n)']' = P^{-1}\hat{C}^{-1}Y(n)$. As a consequence, the following result is established.

Theorem 2: Let Lemma 2 hold and assume $N = p + m$; then there exists an $\epsilon^* > 0$ such that the system (15), with $U(n)$ given by (38), is asymptotically stable for all $\epsilon \in (0, \epsilon^*]$. Moreover, the relations (30), (31) and

$$Y(n) = Y_s(n) + Y_f(n) + 0(\epsilon) \quad (39)$$

hold uniformly for all $n \geq 0$.

V. Conclusion

The block processing method has been applied to singularly perturbed discrete-time systems. The slow and fast subsystems have been expressed in the form of block state-space structures. For these structures, slow and fast controls have been designed based upon state and output measurements. An extra compensating control is proposed which leads to an exact compensation of the effects of the parasitics. Moreover, it is shown that by applying the proposed controls to the full system, the slow dynamics remain slow while the fast dynamics become faster. The block processing technique offers several other prominent advantages such as reduced finite word length effects [3], reduced coefficient sensitivity, and increased data throughput rate. In addition, the parallel nature of this scheme is ideally suited for implementation in a multiprocessor environment [7]. The multirate implementation of the technique proposed in this paper is under investigation.

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Deterministic Control for a New Class of Uncertain Dynamical Systems

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Abstract—**We study the problem of stabilization of nonlinear multivariable uncertain systems. We show that for a *nonlinear* plant with *nonlinear* input, a class of stabilizing continuous-type controllers can be constructed.**

I. Introduction

When modeling a "real" system, one usually does not have, or cannot obtain, an "exact" model. The model usually contains uncertain elements, for example, uncertainties due to parameters, constant or varying, which are unknown or imperfectly known, or uncertainties due to unknown or imperfectly known inputs into the system.

The deterministic approach, used in [1]–[6] for the design of stabilizing feedback controllers for linear multivariable uncertain systems, has been extended by Corless and Leitmann [7] for a very general class of nonlinear multivariable systems. In all the cases above, the essential knowledge required for this design is the *size* of those uncertain elements. No statistical information about the uncertainty is assumed. For the nonlinear case, Corless and Leitmann [7] and Barmish *et al.* [8] showed that a nonlinear plant, stable or unstable, with *linear* input, can be stabilized by a feedback controller. In this paper, we demonstrate that a nonlinear plant with *nonlinear* input can still be stabilized. A continuous-type controller for this purpose is also synthesized. We note that, although [9] considers a more general class of uncertain systems, the controllers there can depend discontinuously on the state.

II. Definitions and Notations

A function $g: \mathcal{K} \times \mathbf{R} \to \mathbf{R}^p$, $\mathcal{K} \subset \mathbf{R}^l$, is Caratheodory if and only if for each $t \in \mathbf{R}$, $g(\cdot, t)$ is continuous; for each $x \in \mathcal{K}$, $g(x, \cdot)$ is Lebesgue measurable; and, for each compact subset $\mathcal{D}$ of $\mathcal{K} \times \mathbf{R}$, there exists a Lebesgue integrable function $M_{\mathcal{D}}(\cdot)$ such that, for all $(x, t) \in \mathcal{D}$,

$$\|g(x, t)\| \le M_{\mathcal{D}}(t). \tag{1}$$

A Caratheodory function $g: \mathcal{K} \times \mathbf{R} \to \mathbf{R}^p$ is strongly Caratheodory iff it satisfies (1) with $M_{\mathcal{D}}(\cdot)$ replaced by a constant $M_{\mathcal{D}}$. A nonlinear system

$$\dot{x}(t) = g(x(t), t) \tag{2}$$

is globally practically stable iff there exists a neighborhood $\underline{\mathcal{B}}$ of $x = 0$ such that the following properties hold: i) (2) has global existence of solutions [9], [10]; ii) (2) has indefinite continuation of solutions [9], [10]; iii) the solutions of (2) are globally uniformly bounded, i.e., given any compact subset $\mathcal{C}$ of $\mathcal{K}$, there exists $d(\mathcal{C}) \in \mathbf{R}_+$ such that, if $x(\cdot):[t_0, \infty) \to \mathcal{K}$ is any solution of (2) with $x(t_0) \in \mathcal{C}$, then $\|x(t)\| \le d(\mathcal{C})$ for all $t \in [t_0, \infty)$; iv) the solutions of (2) are globally, uniformly, ultimately bounded, i.e., given any neighborhood $\mathcal{B}$ of $x = 0$, $\mathcal{B} \supset \underline{\mathcal{B}}$, and any compact subset $\mathcal{C}$ of $\mathcal{K}$, there exists $\mathcal{T}(\mathcal{C}, \mathcal{B}) \in \mathbf{R}_+$ such that, if $x(\cdot):[t_0, \infty) \to \mathcal{K}$ is any solution of (2) with $x(t_0) \in \mathcal{C}$, then $x(t) \in \mathcal{B}$ for all $t \ge t_0 + \mathcal{T}(\mathcal{C}, \mathcal{B})$; v) (uniform stability about a neighborhood) given any neighborhood $\mathcal{B}$ of $x = 0$, $\mathcal{B} \supset \underline{\mathcal{B}}$, there exists a neighborhood $\mathcal{B}_0$ of $x = 0$ such that, if $x(\cdot):[t_0, \infty) \to \mathcal{K}$ is any solution of (2) with $x(t_0) \in \mathcal{B}_0$, then $x(t) \in \mathcal{B}$ for all $t \in [t_0, \infty)$.

III. Main Results

Consider an uncertain system

$$\dot{x} = f(x, t) + B(x, t)\phi(u, \sigma, t) + B(x, t)e(x, \sigma, t) \tag{3}$$

were $t \in \mathbf{R}$, $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control, $\sigma \in \mathbf{R}^p$ is the time-varying uncertain parameter. We assume that $\sigma(\cdot)$ is Lebesgue measurable and its values lie within a prescribed compact set $\Sigma \subset \mathbf{R}^p$, the functions $f(\cdot)$, $\phi(\cdot)$, and $e(\cdot)$ are Caratheodory, and the function $B(\cdot)$ is strongly Caratheodory.

Theorem: Suppose the uncertain system (3) satisfies the following assumptions.

A.1) There exists a known continuous function $\psi(\cdot): \mathbf{R}_+ \to \mathbf{R}_+$, $\psi(0) = 0$, $\psi(p) > 0$ for $p > 0$, such that

$$\gamma(\|u\|) \le u^T\phi(u, \sigma, t) \tag{4}$$

for all $(u, t) \in \mathbf{R}^m \times \mathbf{R}$, $\sigma \in \Sigma$. Furthermore, $\phi(0, \sigma, t) = 0$.

A.2) There exists a known continuous function $\gamma(\cdot): \mathbf{R}_+ \to \mathbf{R}_+$, $\gamma(0) = 0$, $\gamma(p) > 0$ for $p > 0$, such that

$$\gamma(\psi(q)) \ge q\psi(q) \qquad \text{for all } q \ge 0. \tag{5}$$

A.3) The origin $x = 0$ of the nominal system

$$\dot{x} = f(x, t), \tag{6}$$

$f(0, t) = 0$, is a uniformly asymptotically stable equilibrium point. Moreover, there exist a C^1 Lyapunov function $V(\cdot): \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ and functions $\gamma_i(\cdot)$, $i = 1, 2, 3$, belonging to class $\mathbf{K}$ [12], such that

$$\gamma_1(\|x\|) \le V(x, t) \le \gamma_2(\|x\|) \tag{7}$$

$$\frac{\partial V(x, t)}{\partial t} + \nabla_x^T V(x, t) f(x, t) \le -\gamma_3(\|x\|). \tag{8}$$

Then the control $u(t) = p(x(t), t)$, with $p(x, t)$ described below, renders system (3) globally practically stable. Here the function $p(\cdot): \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^m$, strongly Caratheodory, is such that for $\epsilon > 0$,

$$p(x, t) = \frac{-\mu(x, t)}{\|\mu(x, t)\|}\psi(\rho(x, t)) \qquad \text{if } \|\mu(x, t)\| > \epsilon \tag{9}$$

$$p(x, t)\|\alpha(x, t)\| = -\|p(x, t)\|\alpha(x, t) \qquad \text{if } \|\mu(x, t)\| \le \epsilon \tag{10}$$

where

$$\alpha(x, t) = B^T(x, t)\nabla_x V(x, t)$$

$$\mu(x, t) = \alpha(x, t)\rho(x, t)$$

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