Categoricity of Partial Logics

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Timothy Smiley [3] invites us to reconsider the way we think of a logical calculus. He says: "In practice, logical calculi are identified with their consequence relations; thus the test for the adequacy of an axiomatization is whether it succeeds in equating deducibility ($\vdash$) with consequence ($\models$)." (p. 6) He contends that the practice is justified "only if specifying the consequence relation is sufficient to determine the entire sentential output of the semantics". (p. 7) Smiley shows that the standard consequence relation for the propositional logic (as discussed in virtually every elementary course in logic) does not determine the sentential output of the semantics. To do so, it would have to determine all of the logical relationships among the sentences of the calculus. It would tell us "which sentences are the logical truths, which pairs are equivalents, contraries, subcontraries or contradictories..." (p. 7) But, as Smiley shows, the standard consequence relation for the propositional calculus does not tell us that the sentences p and -p are contraries, for example. That is, the consequence relation for the single-conclusion propositional calculus is not categorical.

So, there is a "categoricity problem" for a logical calculus that is solved by providing a consequence relation that specifies the semantic output of the calculus. Smiley [3] solves the categoricity problem for total-valuation logics—logics in which the semantic output consists of values that are designated or undesignated. Ian Rumfitt [2] extends Smiley's work by solving the categoricity problem for multiple-conclusion, partial-valuation logics—logics that permit truth-value gaps. One purpose of the present paper is to extend Smiley's work by solving the categoricity problem for single-conclusion, partial-valuation logics. Categorical deducibility will also be discussed and illustrated.

My general goal is to encourage the development of "judgement logics", which have not received the attention they deserve.

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1. An Illustration of the Categoricity Problem

Consider a language with only two sentences: A and B. A semantics for the language is determined by the array

	A	B
c_1	t	t
c_2	f	t

provided there is also an understanding about which, if any, of the values t and f are designated. Smiley [3] says "The semantics of a calculus produces a classification of every assignment of truth-values (across all the sentences of the calculus simultaneously) as being either "possible" or "impossible"". Call this classification the '*sentential output* of the calculus' (p. 225). So, given the semantics indicated by the above array, the classifications $\langle A/t, B/t\rangle$ and $\langle A/f, B/t\rangle$ are possible (p-classifications); but the classifications $\langle A/t, B/f\rangle$ and $\langle A/f, B/f\rangle$ are impossible (i-classifications).

Given the standard definition of semantic consequence for single-conclusion calculi, $X \models y$ iff for every p-classification of sentences in X gets an undesignated value or the sentence y gets a designated value. Assume that t is the designated value. So, for the above semantics, call it S_1, $\models_{S_1} = \{\langle\varnothing, B\rangle, \langle\{A\}, A\rangle, \langle\{A\}, B\rangle, \langle\{B\}, B\rangle, \langle\{A, B\}, A\rangle, \langle\{A, B\}, B\rangle\}$.

Let us form semantics S_2 by deleting classification c_1 from the above array. Assuming that t is the designated value, it follows that $\models_{S_1} = \models_{S_2}$. Note, for example, that $\varnothing \models_{S_1} B$ and $\varnothing \models_{S_2} B$; and $\{B\} \not\models_{S_1} A$ and $\{B\} \not\models_{S_2} A$. This means that neither $\models_{S_1}$ nor $\models_{S_2}$ is a categorical consequence relation. For, though the two consequence relations have the same extension, the semantics that produces them are very different. Given semantics S_1, A and B are not contraries, and A is contingent; given semantics S_2, A and B are contraries, and A is necessarily false. So, there is a categoricity problem—that of defining consequence so that if the semantics for a language is distinct from another semantics for the language then the corresponding consequence relations are distinct. The above example parallels an example Smiley [3] uses to call attention to the categoricity problem. Consider this array for propositional calculus sentences:

	A	$-A$	$(A \& B)$	...
c	t	t	T	(t)
d	f	t	f	...
e	t	f	f	...
...				

If the semantics is altered by deleting classification c the resulting semantics is very different from the original. A and $-A$ become contraries. Still, the distinct arrays generate the same consequence relation, assuming that t is the designated value.

2. Smiley's Force Solution for Sc-tv Logics

Smiley [3] solves the categoricity problem for single-conclusion (sc), total-valuation (tv) logics by using two forces: strong assertion and strong rejection. We use this Frege-inference he cites to identify these forces: 'If he was not in Berlin, he did not commit the murder. He was not in Berlin. So he did not commit the murder.' Smiley, unlike Frege, thinks that the second premise and the conclusion may be thought of as denials (which, like assertions) are forces. They agree that the first premise is an assertion with no embedded forces. Using $\oplus$ for strong assertion and $\ddagger$ for strong denial, Smiley, in disagreement with Frege, would agree that the above argument may be presented in this form: $\oplus(\neg B \supset \neg M)$. $\ddagger B$. So $\ddagger M$.[1] Frege, as interpreted by Smiley, insists that there is no distinction to be made between $\ddagger B$ and $\oplus\neg B$. And, according to Smiley, Frege does not think that the notion of denial as a force should come into play when discussing logical consequence.

Consider a language with sentences A, B, ... Let $\oplus s$ and $\ddagger s$ be *judgements* iff s is a sentence. A p-classification *satisfies* $\oplus s$ iff it gives s a designated value and *satisfies* s iff it gives s an undesignated value. If J is a set of judgements and j is a judgement then $J \vDash^S j$ iff there is no p-classification that satisfies every member of J and does not satisfy j.

The consequence relation $\vDash^S$ ('S' for 'Smiley') for "judgement logic" shares some of the features of the normal consequence relation for "sentence logic". We mention some facts and provide familiar names in parentheses of principles that are exemplified. Set brackets are omitted. $\oplus A \vDash^S \oplus A$ (reflexivity); if $\ddagger A \vDash^S \oplus B$ then $\ddagger A, \oplus C \vDash^S \oplus B$ (dilution); if $\ddagger A \vDash^S \oplus B$ and $\oplus B \vDash^S \oplus C$ then $\ddagger A \vDash^S \oplus C$ (transitivity). And this is an instance of "reductio": If $\oplus A \vDash^S \oplus B$ and $\oplus A \vDash^S \ddagger B$, then $\varnothing \vDash^S \ddagger A$.

Theorem 1 (Smiley) $\vDash^S$ *is categorical.*

Proof: Given a semantics, there are only two kinds of classifications we need to consider. Suppose c is a classification that assigns designated values to every sentence. Then c is a p-classification iff $\oplus(X - \{x\}) \nvDash^S \ddagger x$, where X is the universal set of sentences and $x \in X$. Suppose c is a classification that assigns an undesignated value to sentence y. Then $\oplus X \cup \ddagger(Y - \{y\}) \nvDash^S \oplus y$, where $X \cup Y$ is the universal set of sentences, each sentence in X is designated, and each sentence in Y is undesignated.

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[1] Smiley uses an asterisk instead of our double dagger for strong rejection and uses no special symbol for strong assertion. Given our interest in distinguishing two kinds of rejection and two kinds of assertion, it is helpful to deviate from his use of symbols.

So, for example, consider the array mentioned above that we used to show that $\vDash_{S_1}$ and $\vDash_{S_2}$ are not categorical. $\langle A/t, B/t\rangle$ is a p-classification for semantics S_1 but not for semantics S_2. Note that $\oplus A \nvDash^{S}_{S_1} \ddagger B$, but $\oplus A \nvDash^{S}_{S_2} \ddagger B$. For both S_1 and S_2 $\langle A/f, B/f\rangle$ is an i-classification. As indicated by the proof, $\ddagger A \vDash^{S}_{S_1} \oplus B$ and $\ddagger A \vDash^{S}_{S_2} \oplus B$. Since $\vDash^{S}_{S_1}$ and $\vDash^{S}_{S_2}$ determine the semantic output of S_1 and S_2, respectively, the two consequence relations are categorical.

3. Rumfitt's Force Solution for Mc-pv Logics

To illustrate the notion of a partial-valuation logic we modify an example used in Section 1 above. Consider a language with only two sentences: A and B. A semantics for the language is determined by the array

	A	B
c_1	t	t
c_2	f	t
c_3	f	

where t is the designated value. This is not a total-valuation logic since the p-classification c_3 assigns neither a designated nor an undesignated value to B. That is, the semantics contains truth-value gaps.

Rumfitt extends Smiley's [3] notion of semantic consequence to inferences with sets of judgements as conclusions. So, given a semantics and sets J and K of judgements, $J \vDash^{R} K$ ('R' for 'Rumfitt') iff there is no p-classification which satisfies every member of J and satisfies no member of K.

Let us refer to the above semantics as S_3, where t is the designated value. Then, for example, $\{\oplus A, \ddagger B\} \vDash^{R}_{S_3} \varnothing$, and $\varnothing \vDash^{R}_{S_3} \{\ddagger A, \oplus B\}$.

But $\{\oplus A\} \nvDash^{R}_{S_3} \{\oplus B, \ddagger B\}$, and $\{\ddagger A\} \nvDash^{R}_{S_3} \{\oplus B, \ddagger B\}$.

As another illustration of the notion of semantic consequence, notice that a semantics in which there are no p-classifications does not generate the same consequences as a semantics in which there is exactly one p-classification that assigns no value to either A or B. For the former, but not the latter, $\{\oplus A, \ddagger A, \oplus B, \ddagger B\}$ is a consequence of the empty set.

Given any semantics for a language with sentences A and B,
$\oplus A \vDash^{R} \{\oplus A, \ddagger B\}$ (an instance of overlap).

And if $\oplus A \vDash^{R} \oplus B$ and $\oplus A \vDash^{R} \ddagger B$, then $\varnothing \vDash^{R} \ddagger A$, provided that $\varnothing \vDash^{R} \{\oplus A, \ddagger A\}$ (an instance of "qualified" *reductio*).

Theorem 2 (Rumfitt) $\vDash^{R}$ *is categorical.*

Proof: Let a classification c assign designated, undesignated, and no values to sentences in X, Y, and Z, respectively. Then c is a p-classification iff $\oplus X$, $\ddagger Y \nvDash^{R} \oplus Z, \ddagger Z$.

So, for example, consider p-classification c_3 for semantics S_3 above. Note that $\ddagger\{A\} \not\models \{\oplus B, \ddagger B\}$.

4. A Force Solution for Sc-pv Logics

Oddly, though Rumfitt [2] calls attention to materials that may be used to provide a direct and more interesting extension of Smiley's [3] solution of the categoricity problem for single-conclusion, total-valuation logics, he does not do so. Rumfitt informally distinguishes two kinds of assertion. By using two kinds of assertion and two kinds of denial we can solve the categoricity problem for single-conclusion, partial-valuation logics.

Rumfitt [2] says that by answering 'No' to the question 'Is the present King of France bald?' you may be indicating that 'the relevant proposition is being rejected as false. And this kind of rejection ... differs from the rejection of a proposition as merely being not true. If you do not believe that there is a present King of France, you will not wish to answer 'No' to the question ... [but you will] certainly refuse to give the answer 'Yes' ...' (p. 226). We have used ‡ for the former (strong) notion of rejection. For the latter (weak) notion of rejection we shall use †. Corresponding to these two notions of rejection are strong assertion (⊕) and weak assertion (+): asserting as true, and asserting as not false, respectively.[2]

Given the above distinctions, we extend the notions of judgement and single-conclusion consequence in the natural way. Let $\mathcal{F}$ range over forces in $\{\oplus, +, \ddagger, \dagger\}$. $\mathcal{F}s$ is a judgement iff s is a sentence. A classification c satisfies $+s$ iff it does not assign an undesignated value to s; and a classification c satisfies $\dagger s$ iff it does not assign a designated value to s. Let J be a set of judgements and let j be a judgement. $J \models j$ for a pv-semantics iff there is no p-classification that satisfies every member of J and does not satisfy j.

Suppose we have a pv-semantics for a language with sentences A, B, and C. $\oplus A \models +A$, and $\ddagger A \models \dagger A$. If $+A \models \oplus A$, $\oplus A \models \ddagger B$, and $\ddagger B \models \dagger C$, then $+A \models \dagger C$ (transitivity). And if $\oplus A \models \oplus B$ and $\oplus A \models \dagger B$, then $\varnothing \models \dagger A$ (reductio).

Theorem 3 $\models$ *is categorical.*

Proof: Given a semantics for a language, for any classification c let X, Y, and Z be the sets of sentences that are designated, undesignated, and neither designated nor undesignated, respectively. There are only three cases to consider.

[2] Rumfitt uses 'internal rejection' and 'external rejection' to refer to strong and weak rejection, respectively. Woodruff [4] suggests 'hedged' (and 'weak') may be used when referring to the two weak forces.

1) Suppose that c assigns a designated value to sentence x. Then c is a p-classification iff $\oplus(X - \{x\})$, $\ddagger Y$, $+Z$, $\dagger Z \nvDash \dagger x$. 2) Suppose that c assigns an undesignated value to sentence y. Then c is a p-classification iff $\oplus X$, $\ddagger(Y - \{y\})$, $+Z$, $\dagger Z \nvDash +y$. 3) Suppose that c assigns no value to sentence z. Then c is a p-classification iff $\oplus X$, $\ddagger Y$, $+Z$, $\dagger Z \nvDash \oplus z$.

5. Categorical Deducibility for Bochvar Logic

Smiley [3] captures the propositional calculus syntactically with his definition of deducibility for single conclusions. Rumfitt [2] captures Bochvar logic syntactically by using multiple conclusions. We will define deducibility for single conclusions to capture Bochvar logic in a simpler way. Moreover, our Jeffrey-style proofs of soundness and completeness are simpler than Rumfitt's Henkin-style proofs.

Consider a language with sentences A, B, ... formed in the standard way by using unary connectives $\neg$ and **T** and binary connective $\wedge$. The p-classifications are generated by using the following tables:

$\neg$	
T	f
F	t

T	
t	t
	f
f	f

$\wedge$	t	f
t	t	f
f	f	f

So, for example, if A is assigned no value then $\neg A$ is assigned no value but $\mathbf{T}A$ is assigned f.

We use $\vDash_B$ to refer to the semantic consequence relation determined by the p-classifications, where t is the designated value.

We follow Jeffrey [1] and define deducibility by using trees. Let $cd\oplus = \dagger$, $cd+ = \ddagger$, $cd\ddagger = +$, and $cd\dagger = \oplus$. Our tree rules are:

1. From $\mathcal{F}\neg A$ infer $cd\mathcal{F}A$.

2. From $\oplus\mathbf{T}A$ ($+\mathbf{T}A$) infer $\oplus A$.

3. From $\ddagger\mathbf{T}A$ ($\dagger\mathbf{T}A$) infer $\dagger A$.

4. From $\oplus(A \wedge B)$ infer $\dfrac{\oplus A}{\oplus B}$.

5. From $+(A \wedge B)$ infer $\dfrac{\oplus A}{\oplus B} \mid \dfrac{+A}{\dagger B} \mid \dfrac{+B}{\dagger B}$.

6. From $\ddagger\,(A \wedge B)$ infer $\dfrac{\oplus A}{\ddagger B} \mid \dfrac{\ddagger A}{\oplus B} \mid \dfrac{\ddagger A}{\ddagger B}$.

7. From $\dagger(A \wedge B)$ infer $\dagger A \mid \dagger B$.

8. From $\oplus A$ and $\ddagger A$ ($\dagger A$) infer **x.**

9. From $+A$ and $\ddagger A$ infer **x.**

By definition $J \vdash_B \mathcal{J}s$, where J is a finite set of judgements, iff a finished tree with the members of $J \cup cd\mathcal{J}s$ as its initial list has no open paths.

So, for example, $\{\oplus A\} \vdash_B +A$, since the tree, with the single path—$\oplus A$, $\ddagger A$, **x**—is closed.

Theorem 4 (soundness and completeness) $\vDash_B = \vdash_B$.

Proof: Imitate Jeffrey's [1] proof. Show that the tree rules are sound and complete. Note, for example that if a p-classification satisfies $\oplus\neg A$ it satisfies $\dagger A$, and vice versa.

6. Woodruff's Judgement Logic

After the LOGICA '98 conference, I became aware of Peter Woodruff's valuable work on judgement, partial-valuation logics. To my knowledge Woodruff [4] gives the first judgement logic that uses these two forces: strong assertion and weak assertion. In Woodruff's judgement logic (using the above terminology for forces and non-Bochvar interpretations of connectives) $\{\oplus A, +(A \supset B) \vDash^W +B)\}$ (p. 127) and $+(T(A \supset Tu) \supset Tu) \vDash^W +A$ (p. 132).

Given our present concerns, it is natural to ask whether $\vDash^W$ ('W' for 'Woodruff') is categorical.

Theorem 5 $\vDash^W$ is not categorical.

Proof: Consider a language with one sentence A. Let S' be a semantics with one p-classification $\langle A\ /f\rangle$ and let S'' be a semantics with the additional p-classification $\langle A\ /t\rangle$. Note that $\vDash^W_{S'} = \{\langle\{\oplus A\}, \oplus A\rangle, \langle\{\oplus A\}, +A\rangle, \langle\{+A\}, \oplus A\rangle, \langle\{+A\}, +A\rangle, \langle\{\oplus A, +A\}, \oplus A\rangle, \langle\{\oplus A, +A\}, +A\rangle = \vDash^W_{S''}$.

7. A Simpler Force Solution for Sc-pv Logics

Consider the following dialogue.

A: Is Goldbach's conjecture true?
B: I cannot say.
A: Is it false?
B: I cannot say.
A: Is it one or the other?
B: Of course, though there are some anti-realist lunatics who disagree with me.

B's assertion cannot be expressed by using either $\oplus$ or +. We use $\sqcup$ to refer to the kind of assertion B makes.[3]

Let $\mathcal{F}$ range over $\{+, \dagger, \sqcup\}$ and let a p-classification satisfy $\sqcup s$ iff it assigns either a designated value or an undesignated value to s. Define $\models'$ in the natural way.

So, for example, if $\varnothing \models' \sqcup A$ then $+A \not\models' \dagger A$.

Theorem 6 $\models'$ *is categorical.*

Proof: Given a semantics for a language, for any classification c let X, Y, and Z be the sets of sentences that are designated, undesignated, and neither designated nor undesignated, respectively. There are only three cases to consider. 1) Suppose that c assigns a designated value to sentence x. Then c is a p-classification iff $+X, \sqcup X, \dagger Y, \sqcup Y, +Z, \sqcup Z \not\models' \dagger x$. 2) Suppose that c assigns an undesignated value to sentence y. Then c is a p-classification iff $+X, \sqcup X, \dagger Y, \sqcup Y, +Z, \sqcup Z \not\models' +y$. 3) Suppose that c assigns no value to sentence z. Then c is a p-classification iff $+X, \sqcup X, \dagger Y, \sqcup Y, +Z, \sqcup Z \not\models' \sqcup z$.

Problems regarding the use of other forces to define categorical consequence are left open.

[3] Recently, Peter Woodruff mentioned this notion of assertion to me. As far as I know, it has not been discussed in published work on judgement logics.

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