Thesis

Towards a General Theory of Erdős-Ko-Rado Combinatorics

Submitted by

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Abstract

Towards a General Theory of Erdős-Ko-Rado Combinatorics

In 1961, Erdős, Ko, and Rado proved that for a universe of size $n \geq 2k$, a family of k-subsets whose members pairwise intersect cannot be larger than $\binom{n-1}{k-1}$. This fundamental result of extremal combinatorics is now known as *the EKR theorem for intersecting set families*. Since then, there has been a proliferation of similar EKR theorems in extremal combinatorics that characterize families of more sophisticated objects that are largest with respect to a given intersection property. This line of research has given rise to many interesting combinatorial and algebraic techniques, the latter being the focus of this thesis.

Algebraic methods for EKR results are attractive since they could potentially give rise to a unified theory of EKR combinatorics, but the state-of-the-art has been shown only to apply to sets, vector spaces, and permutation families. These categories lie on opposite ends of the stability spectrum since the stabilizers of sets and vector spaces are large as possible whereas the stabilizer of a permutation is small as possible. In this thesis, we investigate a category that lies somewhere in between, namely, *the perfect matchings of the complete graph*. In particular, we show that an algebraic method of Godsil's can be lifted to the more general algebraic framework of Gelfand pairs, giving the first algebraic proof of the EKR theorem for intersecting families of perfect matchings as a consequence. There is strong evidence to suggest that this framework can be used to approach the open problem of characterizing the maximum t-intersecting families of perfect matchings, whose combinatorial proof remains illusive. We conclude with obstacles and open directions for extending this framework to encompass a broader spectrum of categories.

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CHAPTER 1

Introduction

Erdős-Ko-Rado or EKR-type combinatorics is a branch of extremal combinatorics that focuses on how large a collection of finite objects can be subject to a given intersection restriction. The quintessential example is *the EKR theorem for intersecting set families*, a classic result of the field that gave it its namesake.

Theorem 1 ([1] Erdős, Ko, Rado 1961). *Let $\mathcal{F}$ be a family of k-subsets of an n element universe such that its members pairwise t-intersect. If $n \geq 2k$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Equality holds if and only if all the members of $\mathcal{F}$ have one element in common.

Theorem 1 is actually a special case of their more general result about *t-intersecting* set families, that is, families of k-subsets such that $|S \cap T| \geq t$ for all $S, T \in \mathcal{F}$. This result is known as *the full EKR theorem for intersecting set families.*

Theorem 2 ([1] Erdős, Ko, Rado 1961). *Let $\mathcal{F}$ be a family of k-subsets of an n element universe such that its members pairwise intersect. If $n \geq 2k$, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}$$

Equality holds if and only if all the members of $\mathcal{F}$ have a fixed set of t elements in common.

Over the years there has been a proliferation of EKR-type results that characterize families of other more sophisticated finite objects that are largest with respect to a given t-intersection property.[1] To get a better idea of what EKR combinatorics entails, we give a list of a few full EKR theorems for different categories.

THEOREM 3 ([2] Frankl, Wilson 1986). *If $\mathcal{F}$ is an intersecting family of k-dimensional subspaces of an n-dimensional vector space over the q-element field, i.e.,* $\dim(S \cap T) \geq 1$ *for any $S, T \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}$$

subject to $n \geq 2k$. Equality holds if and only if every subspace in $\mathcal{F}$ contains a common nonzero vector except the case $n = 2k$.

THEOREM 4 ([3] Frankl, Tokushige 1999). *Let $\mathcal{F}$ be a t-intersecting family of length-n sequences of integers from $\{1, 2, \cdots, q\}$, i.e., $x_i = y_i$ for t or more indices for any $x, y \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq q^{n-1}$$

subject to $q \geq t + 1$. Equality holds if and only if every sequence in $\mathcal{F}$ contains a common fixed subsequence of length t.

THEOREM 5 ([4] Ellis, Friedgut, Pilpel 2011). *Let $\mathcal{F}$ be a family of t-intersecting permutations on n elements, i.e., $\pi(i) = \sigma(i)$ for t or more symbols for any $\pi, \sigma \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq (n-t)!$$

[1] There are other intersection properties that are often considered in EKR combinatorics (e.g. cross-intersecting, partially t-intersecting, exact-t-intersecting). We shall only consider the t-intersection property.

for sufficiently large n. Equality holds if and only if $\mathcal{F}$ is a t-coset of S_n.

In the theorems above, the families that meet the bound with equality are the so-called *trivially t-intersecting families.* Intuitively, these families are the obvious candidates that give a high lower bound on the size of a largest t-intersecting family. A central theme of EKR combinatorics is showing that this lower bound is in fact an upper bound, more specifically, that the trivially t-intersecting families are the only extremal families. Perhaps surprisingly, proving this is the most involved part of (full) EKR results, so it is natural to strive towards a general method for showing that the trivially t-intersecting families are the only largest t-intersecting families.[2]

A well-studied special case of Theorem 5 that attracted a lot of attention is *the EKR theorem for permutations*, first proven by Cameron and Ku in [5] using combinatorial arguments reminiscent of Erdős, Ko, and Rado's proof of Theorem 1.

Theorem 6. *Let $\mathcal{F}$ be a family of intersecting permutations on n elements, i.e., $\pi(i) = \sigma(i)$ for some $i \in \{1, \cdots, n\}$ for any $\pi, \sigma \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq (n-1)!$$

. Equality holds if and only if $\mathcal{F}$ is the coset of the stabilizer of a point.

There are many clever proofs of this theorem [5–7], but most do not seem general enough to apply to other categories of finite objects. One notable exception is a proof by Godsil and Meagher given in [8] where they use a proof technique known henceforth as *the module method* [9]. Below is an outline of their approach, the details of which will be discussed later.

[2] With a "for sufficiently large n" grain of salt in some cases.

(1) Define a non-intersection graph Γ over a collection of objects.

(2) Show that a maximum independent set S of this graph meets the clique/co-clique bound or the ratio bound with equality.

(3) Show v_S lives in the direct sum of the trivial and "standard" module where v_S is the characteristic vector of a maximum independent set S of Γ.

(4) Show that $v_S = v_{ij}$ where v_{ij} is the 0/1 characteristic vector of some trivially intersecting family.

This method seems promising since it has been shown to apply to a wide variety of categories of finite objects such as sets [10], vector spaces [10], and permutations [8]. In [8], Godsil and Meagher leave the viability of the module method over perfect matchings of K_{2n} as an open question.

Our main result is an affirmative answer to this question, giving the first algebraic proof of the EKR theorem for perfect matchings. En route, we give a synthesis of terminology and results in a variety of different subjects, namely, the Theory of Association Schemes, Finite Gelfand Pairs, and Symmetric Function Theory. Many of these connections have been observed before, but they are scattered throughout the literature and their relevance to extremal combinatorics is understated. We give a cohesive account of these connections as they apply to our result.

To put this work in perspective, it is helpful to view a permutation on n symbols as a perfect matching of the complete bipartite graph $K_{n,n}$, and so EKR results for permutations correspond to the EKR results for perfect matchings of $K_{n,n}$.[3] Our work can be seen as a "non-bipartification" of [8] and another testament to the graph-theoretical adage that *non-bipartite matching is more complicated but just as tractable as bipartite matching.*

[3]The correspondence is easily observed after writing a permutation in array notation.

Many of the “bipartite” objects that arise in algebraic proofs of Theorem 6 are well-studied or have since been recognized as interesting, so it is reasonable to assume that our “non-bipartite” objects may also be of independent interest. For example, the notion of a matching derangement is central to the proof of our main result; however, matching derangements have received hardly any attention in combinatorics, unlike their bipartite counterpart (the permutation derangements). Also, *the matching derangement graph* appears to be analogous to *the permutation derangement graph*, a central object of several permutation-theoretic EKR results that has recently enjoyed some attention outside its EKR milieu [11–13]. We put forth a few conjectures regarding the spectrum of the matching derangement graph that are analogues of known results on the permutation derangement graph. Finally, the non-bipartite analogue of *the Birkhoff polytope* arises in our work, but to our knowledge, it has not been studied whatsoever. We hope this work serves as a proper exhibition of these “non-bipartite” objects.

We conclude with speculation on the more general open question of whether the module method can be used to obtain EKR theorems for 1-factors of complete r-uniform hypergraphs K_r^{rn} on rn vertices. Evidence suggests that the module method could be amenable, but will require a significantly more involved (non-commutative) algebraic component. Finally, it is still open whether the module method can be used to prove full EKR theorems; however, the representation theory of perfect matchings is well-studied to the extent that one may be able to give a harmonic analytic proof similar to [4] of the full EKR theorem for perfect matchings, whose combinatorial proof remains elusive.

CHAPTER 2

Preliminaries

2.1. Perfect Matchings

A *matching* of a graph is a subgraph such that each vertex has degree at most one. A matching is *perfect* if each vertex has degree exactly one. Since all matchings considered in this work are perfect matchings, we refer to a perfect matching simply as a matching. A matching of K_{2n}, the complete graph on $2n$ vertices, can be identified as a partition of $[2n] := \{1, 2, \cdots, 2n\}$ where each member of the partition has size two. We shall refer to the matching $e = 1\ 2|3\ 4|\cdots|2n-1\ 2n$ as the *identity matching*. Let $H_n := \{\sigma \in S_{2n} : \sigma e = e\}$ be the subgroup of S_n that stabilizes the identity matching. It is well-known that H_n is the wreath product $S_2 \wr S_n$ which is isomorphic to the *hyperoctahedral group* of order $2^n n!$, the group of symmetries of the n-hypercube. Since matchings are in one-to-one correspondence with cosets of the quotient $M_{2n} := S_{2n}/H_n$, it follows that the number of matchings of K_{2n} is $\frac{2n!}{2^n n!} = (2n-1)!! = 1 \times 3 \times 5 \times \cdots \times 2n-3 \times 2n-1$.

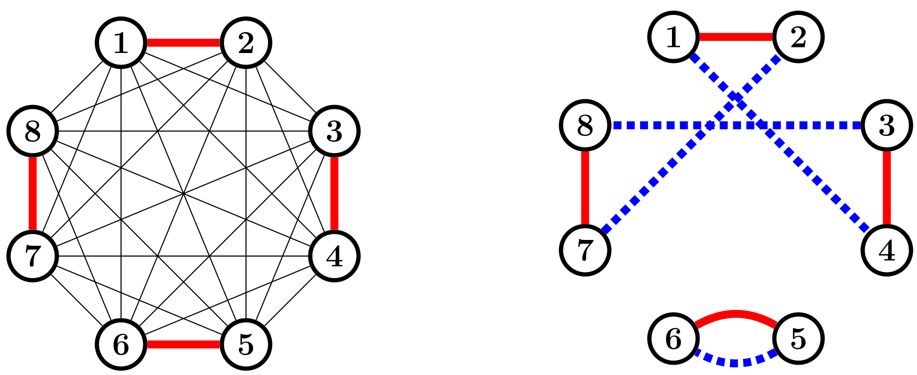

Figure 2.1. On the left, an illustration of K_{2n} and the identity matching e. On the right, the graph $e \cup \sigma e \cong \Gamma_{(n-1,1)}$ where $\sigma \in \{(248), (248)(56), (18234), (18234)(56), (134)(287), (134)(287)(56), (127834), (127834)(56), (173), (173)(56)\}$ for $n = 4$.

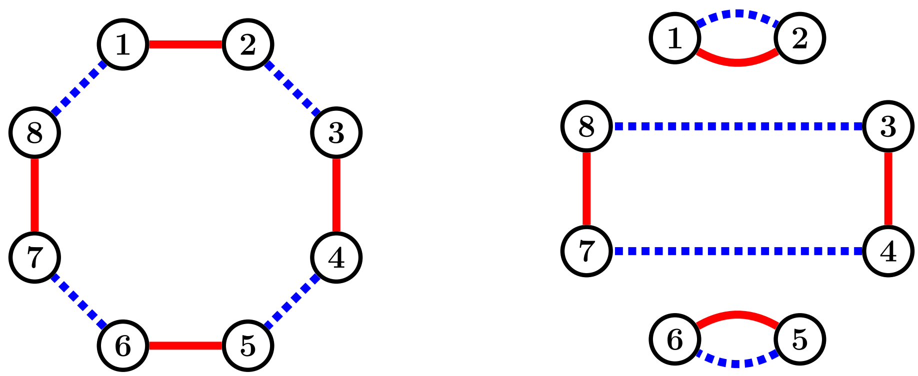

FIGURE 2.2. The matching $m = 2\ 3|4\ 5|6\ 7|1\ 8$ on the left has cycle type $(n) \vdash n$ whereas the matching $m' = 1\ 2|3\ 8|4\ 7|5\ 6$ on the right has cycle type $(2, 1^{n-2}) \vdash n$ where $n = 4$.

For any two matchings $m, m' \in M_{2n}$, let $\Gamma(m, m')$ be the multigraph on $[2n]$ whose edge multiset is the multiset union $m \cup m'$. Clearly $\Gamma(m, m') = \Gamma(m', m)$ and by a theorem of Berge [14], this graph is composed of disjoint cycles of even parity. Let k denote the number of disjoint cycles and let $2\lambda_i$ denote the length of an even cycle. If we order the cycles from longest to shortest and divide each of their lengths by two, we see that each graph corresponds to an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$. For any $\lambda \vdash n$, if there are k parts that all have the same size λ_i, we use λ_i^k to denote the multiplicity. Let $d(m, m') : M \times M \mapsto \mathbf{Part}(n)$ denote this map where $\mathbf{Part}(n)$ is the set of all integer partitions of n. We shall refer to $d(m, m')$ as the *cycle type of* m' *with respect to* m (or vice versa since $d(m, m') = d(m', m)$). If one of the arguments is the identity matching, then we say $d(e, m)$ is *the cycle type of* m. Since $\Gamma(x, y) \cong \Gamma(x', y')$ if and only if $d(x, y) = d(x', y')$, let the graph Γ_λ be a distinct representative from the isomorphism class $\lambda \vdash n$. Illustrations of the graphs $\Gamma_{(n)}$ and $\Gamma_{(2,1^{n-2})}$ are provided in Figure 2.1 where $n = 4$.

Any permutation $\sigma \in S_{2n}$ acts naturally on a matching $m \in M_{2n}$ as follows:

$$\sigma m = \sigma(m_1)\ \sigma(m_2)|\sigma(m_3)\ \sigma(m_4)|\cdots|\sigma(m_{2n-1})\ \sigma(m_{2n})$$

Recall that the action of any permutation $\sigma \in S_n$ on S_n is both regular and faithful. Figure 2.1 illustrates that this is no longer the case when S_{2n} acts on M_{2n}. The action is not regular since there may exist two distinct permutations $\sigma, \rho \in S_{2n}$ such that $\sigma m = \rho m$. The action is not faithful since any union of transpositions of the form $(2i + 1, 2i + 2)$ sends the identity matching to itself. The action is however transitive, since for any two matchings $m, m' \in M$ there exists a $\sigma \in S_{2n}$ such that $\sigma m = m'$. In fact, we have the stronger condition that $\sigma m = m'$ and $\sigma m' = m$ hold simultaneously for some $\sigma \in S_{2n}$, so this is a *generously transitive* action [15]. For any $m, m' \in M_{2n}$, it is easy to obtain such a permutation by considering the cycles of $\Gamma(m, m')$ and picking any permutation $\sigma \in S_{2n}$ that cyclically permutes each cycle, since clearly $\sigma(\sigma m) = m$.

Periodically we will make use of a natural bijection between matchings of $K_{n,n}$ and permutations of S_n that is easily observed after writing an arbitrary permutation, say, $(1\ 2\ 3\ \cdots\ n-1\ n)$, in array notation: $a = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \end{smallmatrix}\right)$. Each row corresponds to a partition class of $K_{n,n}$ and the columns correspond to edges of a matching. Recall that a *derangement* of a permutation $\sigma \in S_n$ is a permutation $\sigma' \in S_n$ that disagrees with σ everywhere, that is, $\sigma(i) \neq \sigma'(i)\ \forall i \in [n]$. Equivalently, $\sigma\pi = \sigma'$ is a derangement of σ if and only if π has no 1-cycle. The set of permutations that have no fixed points are known as the *derangements* of S_n and their size (the subfactorial numbers) can be computed using the following recurrence: $!n := D_n = (n-1)(D_{n-1} + D_{n-2})$ where $D_0 = 1$ and $D_1 = 0$. Under the aforementioned bijection, these numbers also count the number of matchings of $K_{n,n}$ that do not share an edge with an arbitrary matching of $K_{n,n}$.

There is an analogous notion of derangement for matchings that has received hardly any attention in both the combinatorial and algebraic literature. A *(matching) derangement* of

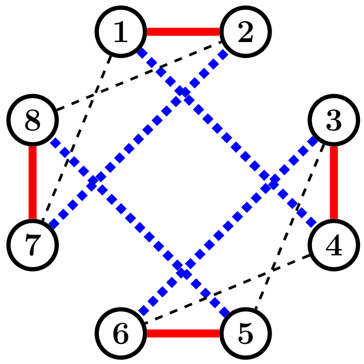

FIGURE 2.3. Three matchings $m_1 = e$, $m_2 = 1\ 4|3\ 6|5\ 8|2\ 7$, and $m_3 = 1\ 7|2\ 8|3\ 5|4\ 6$ each of which are derangements of one another. Note that their cycle types $d(m_1, m_2) = d(m_2, m_3) = (n)$ and $d(m_1, m_3) = (\frac{n}{2}, \frac{n}{2})$ have no 1-cycles.

$m \in M_{2n}$ is a matching m' such that m and m' disagree everywhere, that is, $m \cap m' = \emptyset$. Equivalently, $\sigma m = m'$ is a derangement of m if $d(m, \sigma m)$ has no part of size one (1-cycle). A routine inclusion-exclusion argument shows that the number of derangements D_n^M of an arbitrary $m \in M_{2n}$ can be computed recursively as follows:

$$!!n := D_n^M = 2(n-1)(D_{n-1}^M + D_{n-2}^M)$$

where $D_0^M = 1$ and $D_1^M = 0$.[1] Notice that because S_{2n}/H_n is not a group, there is no well-defined set of derangements for M_{2n} like there is for the symmetric group. Below are the first thirteen values of D_n^M.

$$1, 0, 2, 8, 60, 544, 6040, 79008, 1190672, 20314880, 387099936, 8148296320, 187778717632, \cdots$$

Let K_r^{rn} denote the *hypergraph* on rn vertices such that every subset of r vertices forms a hyperedge. A *1-factor* of K_r^{rn} is a partition of the vertices into subsets of size r. A

[1]To our knowledge, the unary prefix operator (!!) has not been used to express the matching derangement ("double subfactorial") numbers. We feel our choice of notation is natural.

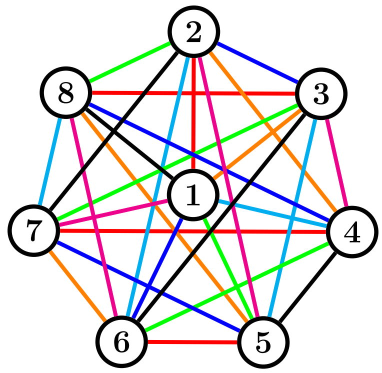

FIGURE 2.4. An illustration of a 1-factorization F of K_{2n} such that $m \cup m' \cong \Gamma_{(n)}$ $\forall m, m' \in F$ where $n = 4$.

1-factorization F of K_r^{rn} is a partition of the hyperedges such that each partition is a 1-factor [16]. The following theorem is due to Baranyai [17].

THEOREM 7. *Every* K_r^{rn} *admits a 1-factorization.*

An *edge coloring* of a graph is a coloring of the edges such that no two edges of the same color are incident to the same vertex. Note that a 1-factorization for $r = 2$ is simply an edge coloring of K_{2n} such that the graph induced by each edge color class is a matching. Moreover, by a theorem of Lucas, every complete graph on $2n$ vertices admits a 1-factorization F such that $\forall m, m' \in F$, $d(m, m') = (n)$ as follows [18]. Let $2n - 1$ of the vertices be the points of a regular $(2n-1)$-gon centered at the origin and place the remaining point at the origin. For each color class, include one edge from the origin to one of the points, and all of the perpendicular edges connecting pairs of polygon points (see Figure 2.4 for an illustration).

Finally, we shall prefer a design-theoretic language for hypergraph matchings and 1-factorizations when $r > 2$. Let $\binom{X}{r}$ denote the *complete design*, that is, the collection of all r-subsets of a ground set X. A *parallelism* of $\binom{X}{r}$ is partition $\Pi = \pi_1, \pi_2, \cdots \pi_k$ of $\binom{X}{r}$ such that each class π_i is a partition of X [19]. It is clear that Baranyai's theorem can be

abc	adg	aei	ceg	acd	aeh	agb	dgh	ade	agi	ahc	ehi	aeg	ahb
def	beh	dhc	fha	egf	cgi	eid	fia	ghf	dhb	gbe	fba	hif	eic
ghi	cfi	gbf	ibd	hib	dfb	hcf	bce	ibc	efc	idf	cdg	bcd	gfd

aid	gib	agh	aic	abe	hbc	ahi	abd	acg	icd	aib	ace	adh	ide
hcg	fca	ibf	gdb	idh	fda	bcf	hce	bei	fea	cdf	idg	cgb	fga
bef	deh	cde	hfe	cgf	egi	deg	ifg	dhf	ghb	egh	bfh	eif	hbc

FIGURE 2.5. A parallelism of the complete design $\binom{X}{3}$ found by Walecki (1896) where $X = \{a, b, c, d, e, f, g, h, i\}$. Each column is a 1-factor of K_3^9.

interpreted as necessary and sufficient conditions for the existence of a parallelism of $\binom{X}{r}$. Figure 2.5 gives an example of a parallelism for $r = 3$.

2.2. Polyhedral Combinatorics

We give a rough overview of basic definitions and elementary properties of convex polytopes. Proofs of the general results below can be found in [20]. We refer the reader to [21] for a better treatment of convex polytopes and their relationship to algebraic graph theory.

Let $P \subset \mathbb{R}^d$ be a convex polytope defined by a set of linear inequalities defined over $\mathbb{R}^d$. Any element of $p \in \mathbb{R}^d$ that satisfies the system of linear equalities is a *point* of P. If a point $p \in P$ satisfies the system of linear inequalities with equality, then it is an *extreme point* or *vertex* of P. The *dimension* of P, denoted $\dim P$, is the dimension of the smallest affine subspace of $\mathbb{R}^d$ containing P. A *face* $F \subseteq P$ is an intersection of P with a hyperplane such that none of the interior points of P lie on the boundary of the halfspace. Each face itself is a polytope. Below is a list of the names of some popular faces of convex polytopes.

- vertices $\leftrightarrow$ 0-dimensional faces of P.
- edges $\leftrightarrow$ 1-dimensional faces of P.
- ridges $\leftrightarrow$ $(\dim P - 2)$-dimensional faces of P.
- facets $\leftrightarrow$ $(\dim P - 1)$-dimensional faces of P.

A *graded poset* is a partially ordered set $(P, \preceq)$ with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. A *graded lattice* L is a graded poset $(P, \preceq)$ in which every pair of elements $p, p' \in P$ has a unique maximal lower bound, called the *meet* $p \wedge p'$, and a unique minimal upper bound, called the *join* $p \wedge p'$. The minimal elements of $L \setminus \{\hat{0}\}$ are the *atoms* of L and the maximal elements of $L \setminus \{\hat{1}\}$ are the *coatoms* of L. A graded lattice L is *atomic* if every element is a join of a set of atoms, and it is *coatomic* if every element is a meet of a set of coatoms. Let $L(P)$ be the set of all faces of P partially ordered by inclusion.

THEOREM 8. *$L(P)$ is a finite, graded, atomic, and coatomic lattice. The meet operation $F \wedge G$ is given by intersection, while the join $F \vee G$ is the intersection of all facets that contain both F and G.*

Due to the above result, we refer to $L(P)$ as *the face lattice of* P. The face lattice of the pyramid is given in Figure 2.6. Theorem 8 tells us that characterizing the *facet-inducing inequalities* of P (those inequalities that uniquely define facets) allows one to completely understand the faces of P. We say that two polytopes P and Q are *combinatorially equivalent* if $L(P) \cong L(Q)$. Our study of polytopes in this work will be up to combinatorial equivalence.

It is well-known that any face of a polytope P can be expressed as the intersection of P with a hyperplane H. In particular, if we let $h \in R^n$, then for each $a \in \mathbb{R}$, the sets

$$H_a = \{x \in \mathbb{R}^n : h^T x = a\}$$

are hyperplanes that partition $\mathbb{R}^n$. If P is a polytope, then there is some $a \in \mathbb{R}$ such that $P \cap H_a \neq \emptyset$. It follows that by finding the maximum and minimum values of a such that $P \cap H_a \neq \emptyset$, we find *parallel* faces of P.

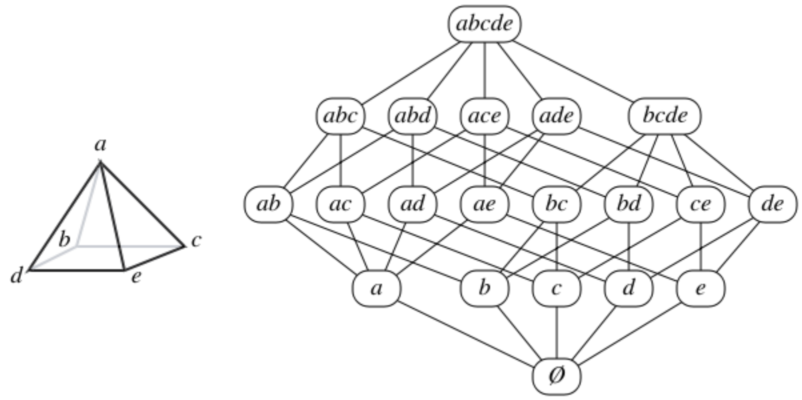

FIGURE 2.6. The pyramid face lattice (courtesy of David Eppstein).

LEMMA 1. *Let P be the convex hull of the rows of a matrix U, then $Uh = z$ and*

$$F_{\min} = \{x \in P : h^T x = z_{\min}\}$$

$$F_{\max} = \{x \in P : h^T x = z_{\max}\}$$

are parallel faces of P. Moreover, if z is a 0/1 vector, then $F_{\min}$ and $F_{\max}$ partition the vertices of P.

We conclude with a description of two convex polytopes that arise in the EKR theorem for permutations and perfect matchings of K_{2n}. These results can be found in [22] and [23].

An $n \times n$ matrix A is *doubly stochastic* if its entries are non-negative real numbers and its rows and columns each sum to 1. The *Birkhoff polytope* is the convex polytope in $\mathbb{R}^{n^2}$ whose points are the doubly stochastic matrices. Below is a set of linear inequalities that defines the polytope.

$$a_{ij} \geq 0; \quad \sum_{i=1}^{n} a_{ij} = 1; \quad \sum_{j=1}^{n} a_{ij} = 1; \quad \forall 1 \leq i, j \leq n.$$

Birkhoff showed that there are $n!$ vertices of this polytope are they are precisely the $n \times n$ permutation matrices. It is clear that the vertices of this polytope correspond to the perfect matchings of $K_{n,n}$, so we shall refer to this polytope $M(K_{n,n})$ as the *perfect matching polytope of* $K_{n,n}$.

Since the rows and columns of any doubly stochastic matrix A must sum to 1, one can deduce the entries a_{nj}, a_{in} $\forall 1 \leq i, j \leq n$ once the entries a_{ij} $\forall 1 \leq i, j \leq n-1$ have been determined, hence the dimension of $M(K_{n,n})$ is $(n-1)^2$. Within the space $\mathbb{R}^{(n-1)^2}$, the remaining n^2 non-negativity constraints of the form $a_{ij} \geq 0$ define the polytope. Birkhoff showed that each of these inequalities induces a unique facet, thus there are n^2 facets of $M(K_{n,n})$.

It is well-known that the edges of $M(K_{n,n})$ correspond to pairs of permutations (π, σ) such that $\pi^{-1}\sigma$ is n-cycle, hence the 1-skeleton (graph) of $M(K_{n,n})$ is the normal Cayley graph $\Gamma(S_n, C_n)$ where $C_n \subseteq S_n$ is the conjugacy class of all n-cycles on n symbols. The correspondence between 1-skeletons of convex polytopes (whose vertices correspond to a complete collection of finite objects) and normal Cayley graphs will be revisited in the final chapter of this work.

We turn our attention to the more general *perfect matching polytope of* G which we denote as $M(G)$. Let x_m be the characteristic vector of a perfect matching m of G, that is, $x_m(e) = 1$ if $e \in m$; otherwise, $x_m(e) = 0$. It is well-known that the vertex set of $M(G)$ is the convex hull of the set $\{x_m \in \mathbb{R}^{|E|} : m \text{ is a perfect matching of } G\}$ and that if $G = K_{2n}$, then $M(K_{2n})$ has $(2n-1)!!$ vertices. For any $S \subseteq V(G)$, let $\delta(S)$ denote the set of edges with exactly one endpoint incident to S. An *odd cut* C of a graph G is a set of edges of the form $\delta(S)$ where $\emptyset \subset S \subset V(G)$. If $|V \setminus S| > 1$ and $|S| > 1$, then we say that C is a *non-trivial odd* *cut*. Let $\mathcal{C}$ be the set of all non-trivial odd cuts of a graph. If G is a connected graph such that every edge belongs to some perfect matching, then $x \in M(G)$ if and only if it satisfies the set of linear inequalities below.

$$x(e) \geq 0 \ \forall e \in E; \quad x(\delta(\{v\})) = 1 \ \forall v \in V; \quad x(C) \geq 1 \ \forall C \in \mathcal{C}$$

THEOREM 9 (Edmonds, Pulleyblank, Lovász 1982). *Let $G = (V, E)$ be a connected graph such that every edge belongs to some perfect matching. Then the dimension of the perfect matching polytope of $M(G)$ is $|E| - |V| + 1 - \beta$ where β is the number of bricks in the brick decomposition of G.*

Since the size of the brick decomposition of a non-trivial clique on an even number of vertices is 1 [23], we have the following as a simple corollary.

COROLLARY 1. $\dim M(K_{2n}) = \binom{2n}{2} - 2n$.

Using highly non-trivial graph theory, the facet-inducing inequalities of $M(G)$ are also characterized in [23]. Their characterization of the facet-inducing inequalities is much too involved to be stated in full, so we refer the interested reader to Theorem 6.2 of [23] for the full graph-theoretical characterization of the facets of $M(G)$. Fortunately, in the special case where G is a non-trivial clique on an even number of vertices, the situation is drastically simplified, leading to the following straightforward corollary of Theorem 6.2 of [23].

COROLLARY 2. *There are $\binom{2n}{2}$ facets of $M(K_{2n})$ each of which corresponds to a facet-inducing inequality of the form $x(e) \geq 0$ where $e \in E$.*

Corollaries 1 and 2 will come into play for the final part of the proof of our main result.

2.3. Finite Group Representation Theory

We refer the reader to [24] for a more thorough exposition of representation theory.

A *representation* of a group G on a vector space V over a field K is a group homomorphism $\rho : G \to GL(V)$ such that $\rho(gh) = \rho(g)\rho(h)$ where $GL(V)$ is the group of $n \times n$ invertible matrices over K. All representations will be defined over the field $K = \mathbb{C}$ of complex numbers unless stated otherwise. A linear subspace $W \subset V$ is *G-invariant* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. The restriction of ρ to a G-invariant subspace is a *subrepresentation* of ρ. A representation ρ is an *irreducible representation* (*irreducible)* or *irrep* for short) if and only if ρ admits no subrepresentations other than V and $\{0\}$.

The *character* of a representation ρ is the function $\chi_\rho : G \to K$ where $g \mapsto \mathrm{Tr}(\rho(g))$. We shall assume that all characters in this work irreducible unless stated otherwise. Characters are *class functions*, that is, they each take a constant value on a given group conjugacy class. More precisely, the set of irreducible characters of G into K form a basis of the K-vector space of all class functions $G \to K$.

Let G be a group of order n and let $\mathbb{C}[G]$ denote the vector space over $\mathbb{C}$ with natural basis $\{\mathbf{g_1}, \mathbf{g_2}, \cdots, \mathbf{g_n}\}$ whose elements are formal linear combinations of the form $\sum_{i=1}^{n} c_i \mathbf{g_i}$. This space can be endowed with the following product:

$$\begin{aligned}\left(\sum_{g\in G} b_g \mathbf{g}\right)\left(\sum_{g\in G} c_g \mathbf{g}\right) &= \sum_{g,h\in G} b_g c_h \mathbf{gh} \\ &= \sum_{g\in G}\sum_{h\in G} b_h c_{h^{-1}g} \mathbf{g}\end{aligned}$$

When $\mathbb{C}[G]$ is endowed with this product, it is often referred to as the *group algebra* of G over $\mathbb{C}$. If G acts naturally on $\mathbb{C}[G]$, that is $h\mathbf{g} \mapsto \mathbf{hg}$, then we obtain the *left regular* or *permutation representation* of G over $\mathbb{C}$, $\pi : G \to GL(\mathbb{C}[G])$. We shall assume that all representations are over $\mathbb{C}$ henceforth.

The permutation representation in general is a reducible representation, which means that it can be decomposed as a direct sum of irreps that are pairwise orthogonal. Moreover, the decomposition of the permutation representation is unique and includes every irrep of G. More formally, we have the following.

THEOREM 10. *Let π be the permutation representation of a group G. Then π is orthogonally equivalent to the following*

$$\pi \cong \bigoplus_{i=1}^{k} \dim(V_i) V_i$$

It will be enlightening to view the group algebra $\mathbb{C}[G]$ as $L^2(\mathbb{C}[G])$, the vector space of complex-valued functions over G. Indeed any formal sum of $\mathbb{C}[G]$ corresponds to a vector $v \in \mathbb{C}^n$, where v corresponds to a complex-valued function f such that $v_i = f(g_i)$. A natural basis for this space is obtained by letting $\mathbf{1}_{g_i}$ be the characteristic vector of g_i and extending by linearity. In this new setting, we obtain the group algebra again by observing that the product defined above is simply *convolution*:

$$(f * g)(x) = \sum_{yz=x} f(y)g(z)$$

where $f, g \in \mathbb{C}[G]$ and $x, y, z \in G$. There is a natural action of G on $L^2(\mathbb{C}[G])$ given by $(xf)(y) = f(x^{-1}y)$ which again gives rise to the permutation representation of G.

2.4. Representation Theory of the Symmetric Group

A few basic representation theoretical results particular to the symmetric group will be needed. We refer the reader to [25, 26] for a more thorough exposition.

Recall that two permutations $\sigma, \rho \in S_n$ belong to the same conjugacy class if and only if they have the same cycle type. The cycle type of any permutation can be expressed as an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum \lambda_i = n$; therefore, conjugacy classes of S_n can be labeled by their integer partition. Any integer partition $\lambda \vdash n$ can be represented as a *Young tableau* T_λ of shape $\lambda \vdash n$ that is composed of n cells (see Figure 2.7). Since shapes and Young tableaux are in one-to-one correspondence, we will often refer to a Young tableau T_λ simply by its shape λ. We use the following conventions for describing the anatomy of Young tableaux. Let $c = (i, j)$ be the cell at row i column j of a Young tableau T_λ.

- The *size* of T_λ is the # of cells and its *height* $H(\lambda)$ is the # of rows.
- The *leg length* of c, $ll(c)$, is the # of cells below c in column j.
- The *arm length* of c, $al(c)$, is the # of cells to the right of c in row i.
- Let $h(c) = al(c) + ll(c)$ be the *hook length* of c.
- T_λ *covers* a shape μ if T_μ is a Young subtableau of T_λ.
- A *rim hook* of T_λ is a Young subtableau T_μ that does not cover $(2, 2)$ and whose removal from T_λ results in a valid Young tableau $T_{\lambda\setminus\mu}$.

Let λ' be the *transpose* shape of λ that one obtains a by reflecting the T_λ along its main diagonal. A *standard Young tableau of shape* $\lambda \vdash n$ is a Young tableau of shape λ whose cells are labeled from $1 \cdots n$ such that the columns and rows are strictly increasing (see Figure 2.7). It is well-known the number of distinct irreps of any group equals the number

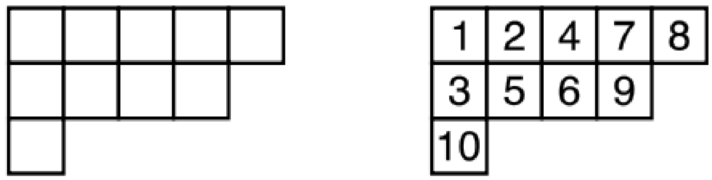

FIGURE 2.7. On the left, the Young tableau of shape $\lambda = (5, 4, 1)$. On the right, one of 288 standard Young tableaux of shape $\lambda = (5, 4, 1)$.

of its conjugacy classes, which implies that the irreps of S_n are in one-to-one correspondence with Young tableaux. Furthermore, the dimension of the irrep of shape λ corresponds to the number of standard Young tableaux of shape λ, which can be obtained using the *hook length formula*.

THEOREM 11 (Hook Length Formula). *The number of standard Young tableau of shape λ is given by the following:*

$$h(\lambda) := \frac{n!}{\prod_{c \in T_\lambda} h(c)}$$

where the product ranges over all cells of T_λ.

Let χ_λ be the irrep of shape λ. In this work we will be concerned with computing the character values of $\chi_\lambda(\pi)$ of where $\pi \in S_n$. Since characters are class functions, we have $\chi_\lambda(\pi) = \chi_\lambda(\mu)$ for all $\pi, \mu \in S_n$ of the same cycle type. In light of this, we define $\chi_\lambda^\alpha := \chi_\lambda(\pi)$ where π is an arbitrary permutation of cycle type α.

The Schur functions s_λ famously have an interpretation as exponential generating functions for the irreducible characters χ_λ of S_n. Let π have cycle type $\mu = (\mu_1, \mu_2, \cdots \mu_m)$.

$$s_\lambda = \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) p_\pi$$

where $p_\pi = p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_m}$ are the *power symmetric functions*. Fortunately, there is a simple combinatorial algorithm for computing the coefficients of this generating function.

Theorem 12 (Murnaghan-Nakayama). *Let $\lambda \vdash n$ and α be a cycle type:*

$$\chi_\lambda^\alpha = \sum_\zeta (-1)^{H(\zeta)} \chi_{\lambda\setminus\zeta}^{\alpha\setminus\alpha_1}$$

where the sum runs over all rim hooks ζ of λ of size α_1.

Corollary 3. *Let $\lambda \vdash n$. If T_λ is not a rim hook, $\chi_\lambda^{(n)} = 0$; otherwise, $\chi_\lambda^{(n)} = -1^{H(T_\lambda)}$.*

2.5. Algebraic Graph Theory and Erdős-Ko-Rado Combinatorics

In this section we overview algebraic graph theory as it applies to EKR combinatorics. We refer the reader to [27] for a more thorough exposition of basic algebraic graph theory and [28] for a better treatment of Cayley graphs and their spectra. We begin by describing a well-known graph-theoretical framework for doing EKR combinatorics.

Let X be a ground set of a finite objects and let $\sim$ be an intersection relation defined with respect to X. Let the *non-intersection graph* be a simple undirected graph $\Gamma(X, \not\sim)$ defined over X such that $x, x' \in X$ are connected if and only if $x \not\sim x'$. Each maximum clique of $\Gamma(X, \not\sim)$ corresponds to a largest family of objects $\mathcal{F} \subseteq X$ such that $x \not\sim x'$ for all $x, x' \in X$. Each maximum independent set of $\Gamma(X, \not\sim)$ corresponds to a largest family of objects $\mathcal{F} \subseteq X$ such that $x \sim x'$ for all $x, x' \in X$. Some of the most obvious examples arise when X is the collection of all k-element subsets of $[n]$.

Example 1. *Let X be the collection of all k-element subsets of $[n]$ and define $x \sim y \Leftrightarrow |x \cap y| \neq 0$. The non-intersection graph $\Gamma(X, \not\sim) \cong KG(n, k)$ is known as the Kneser graph. Its maximum independent sets correspond to largest families of k-subsets such that its members are all pairwise intersecting.*

Example 2. *Let X be the collection of all k-element subsets of $[n]$ and define $x \sim y \Leftrightarrow |x \cap y| \neq k-1$. The non-intersection graph $\Gamma(X, \not\sim) \cong J(n,k)$ is known as the Johnson graph. Its maximum independent sets are the largest families of k-subsets such that the pairwise intersections of its members all have size $k-1$.*

In light of the connection between independent sets and intersecting families, the following bound due to Delsarte and Hoffman has proven to be quite effective. The *characteristic vector* of a set $S \subseteq V$ is a 0/1 vector x of size $|V|$ such that $x(i) = 1$ if $i \in S$; otherwise, $x(i) = 0$.

Theorem 13 ([29] Ratio Bound). *For any weighted k-regular graph Γ on n vertices:*

$$\alpha(\Gamma) \leq n \frac{-\eta_{\min}}{k - \eta_{\min}}$$

where $\eta_{\min}$ is the minimum eigenvalue of Γ and $\alpha(\Gamma)$ is the size of a maximum independent set in Γ. Furthermore, if equality holds then the characteristic vector of a maximum independent set is a linear combination of $\mathbf{1}$ and a $\eta_{\min}$-eigenvector.

The difficulty in applying the ratio bound lies in characterizing the minimum eigenvalue, which is difficult to do for most graphs. Fortunately, the spectra of the Kneser and the Johnson graphs are well-understood [10] and since there exist independent sets of these graphs that meet the ratio bound with equality, it provides elegant proofs of several set-theoretic EKR results. At present, the necessary and sufficient conditions for graphs meeting the ratio bound with equality is very much a mystery. It just so happens that non-intersection graphs arising from EKR combinatorics often meet this bound with equality.

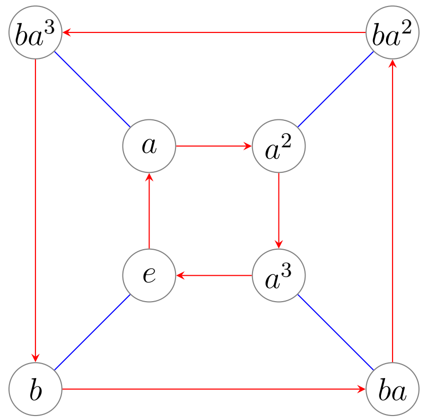

FIGURE 2.8. The directed Cayley graph $\Gamma(D_8, \{a, b\})$

When dealing with ground sets that are more sophisticated than sets (e.g. permutations, partitions, graphs), it becomes exceeding difficult to characterize the spectrum of the corresponding non-intersection graph; however, if the ground set $X = G$ is a finite group and $\sim$ is natural, then $\Gamma(X, \not\sim)$ is often isomorphic to a normal *Cayley graph* of a group G. In this situation, information about the group can shed light on combinatorial and algebraic properties of $\Gamma(X, \not\sim)$.

Let $S \subseteq G$ be a subset of a group G. A *Cayley graph* is a directed graph $\Gamma(G, S)$ defined over the elements of the group such that two elements g, h are connected if and only if $g^{-1}h \in S$. Every Cayley graph is *vertex-transitive*, that is, its automorphism group acts transitively upon its vertices. A Cayley graph $\Gamma(G, S)$ is *normal* if the generating set S is a union of conjugacy classes of G. Figure 2.8 gives an illustration of the Cayley graph over D_8 generated by a rotation and a flip and Figure 2.9 shows that the normal Cayley graph over S_3 generated by transpositions is isomorphic to $K_{3,3}$. Note that any normal Cayley graph defined over the symmetric group is undirected since $\pi \in K \Leftrightarrow \pi^{-1} \in K$ for any conjugacy class K of S_n.

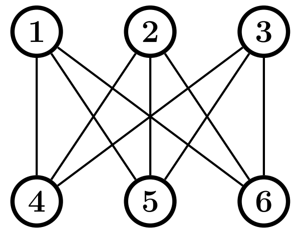

FIGURE 2.9. The normal Cayley graph $\Gamma(S_3, K_{(2,1)})$.

One of the most well-studied normal Cayley graphs in EKR combinatorics naturally arises when $X = S_n$. We say that two permutations $\sigma, \pi \in S_n$ agree on a symbol $i \in [n]$ if $\sigma(i) = \pi(i)$. A family of permutations $\mathcal{F} \subseteq S_n$ is t-intersecting if each pair of its members agree on t or more symbols.

EXAMPLE 3. *Let $X = S_n$ and define $\sigma \sim \pi \Leftrightarrow \sigma$ and π agree on $t-1$ or fewer symbols. The non-intersection graph $\Gamma(S_n, \not\sim) \cong \Gamma(S_n, D_n^t)$ is known as the t-derangement graph. It is a normal undirected Cayley graph whose maximum independent sets correspond to largest families of t-intersecting permutations.*

The t-derangement graph has been central to many permutation-theoretic EKR-type results. When $t = 1$, $\Gamma(S_n, D_n^1)$ is simply referred to as the *(permutation) derangement graph.* Recently, there has been significant effort towards understanding the spectrum of the derangement graph [11–13]. Below we describe a representation-theoretical approach that has been used for understanding spectra of normal Cayley graphs.

Recall that the group algebra $\mathbb{C}[G]$ can be decomposed into orthogonal subspaces each of which corresponds to an irreducible module. In light of this decomposition, operators that act on this space can be understood in terms of invariant subspaces. In particular, for any adjacency matrix A of a Cayley graph $\Gamma(G, S)$ we have $a_{gh} = 1$ if and only if $g^{-1}h \in S$, hence $A = \sum_{s \in S} \phi(s)$ where ϕ is the permutation representation of G. Since each $\phi(s)$ can be decomposed into irreducibles, we have

$$A = \sum_{s \in S} \phi(s) \cong \sum_{s \in S} d_1 \pi_1(s) \bigoplus d_2 \pi_2(s) \bigoplus \cdots \bigoplus d_k \pi_k(s)$$

where $d_i = \dim(\pi_i)$ is the multiplicity of the irreducible representation π_i. This decomposition allows us to analyze the spectrum of A using representation theoretical tools.

THEOREM 14. *Let $\Gamma(G, S)$ be a normal Cayley graph. Then the eigenvalue of $\Gamma(G, S)$ corresponding to the irrep χ_λ is*

$$\eta_\lambda = \frac{1}{\chi_\lambda(1)} \sum_{C \in S} |C| \chi_\lambda^c$$

where C is a conjugacy class of S and $c \in C$ is an arbitrary conjugacy class representative. Furthermore, each eigenvalue η_λ occurs with multiplicity $\chi_\lambda(1)^2$.

The theorem above tells us that each eigenvalue of a normal Cayley graph is completely determined by the character of the irrep. When the character theory associated with the normal Cayley graph $\Gamma(G, S)$ is not unwieldy, then it is feasible to characterize the minimum eigenvalue. Using non-trivial symmetric function theory and representation theory of symmetric group, Renteln did precisely this for the derangement graph.

THEOREM 15 ([11] Renteln 2007). *The minimum eigenvalue of the permutation derangement graph is $-(\frac{!D_n}{n-1})$.*

A simple application of the ratio bound gives an alternate proof of the EKR theorem for permutations. We are now ready to introduce the non-intersection graph that will be essential for our proof of the EKR theorem for perfect matchings.

Definition 1. *Let* $\Gamma_n^M = \Gamma(M_{2n}, D_n^M)$ *be the matching derangement graph defined such that two matchings* $m, m' \in M_{2n}$ *are adjacent if and only if* $d(m, m')$ *has no part of size one.*[2]

In Chapter 5 we give a thorough examination of this graph, but for now, observe that Γ_n^M is not a Cayley graph since S_{2n}/H_n is not a group. It appears that we are far from being able to use any of the aforementioned group representation-theoretic tools; however, if we move to the more general *theory of association schemes*, we will encounter a lesser-known generalization of Theorem 14 that will serve as our representation-theoretic tool for understanding the spectrum of Γ_n^M.

A nice characterization of the eigenvalues of Γ^M is currently open; however, the following combinatorial technique can be used to prove that certain values must be eigenvalues of Γ^M. A partition $\pi = \{C_1, C_2, \cdots, C_m\}$ of the vertices of a graph G is *equitable* if for every pair of indices $i, j \in \{1, 2, \cdots, m\}$ there is a nonnegative integer b_{ij} such that each vertex $v \in C_i$ has exactly b_{ij} neighbors in C_j, regardless of the choice of v. In this situation, the quotient G/π obtained by contracting each partition class C_i to a single vertex is a nonsimple multigraph whose adjacency matrix (quotient matrix) is defined as $B_\pi := (b_{ij})$. The utility of equitable partitions is made obvious by the following lemma.

Lemma 2. *If* $\pi = \{C_1, C_2, \cdots, C_m\}$ *is an equitable partition of* G *with quotient matrix* B_π*, then the spectrum of* B_π *is a subset of the spectrum of* G*.*

Later, we will use this lemma to show that $-\frac{D_n^M}{2n-1}$ is always an eigenvalue of Γ_n^M. We will use the ratio bound to prove a new EKR-type theorem for matchings, but the following theorem, essentially due to Delsarte, will be of central importance for our main result.

[2] We suppress the subscript when n is clear from context.

Theorem 16 ([8, 29] Clique/co-Clique Bound). *Let $\mathcal{A}$ be an association scheme on n vertices and let Γ be the union of some of the graphs in the scheme. If C is a clique and S is an independent set in Γ, then*

$$|C||S| \leq n$$

If $|C||S| = n$ and x and y are the respective characteristic vectors of C and S, then

$$x^T E_j x y^T E_j y = 0 \ \forall j > 0.$$

This gives rise to the following simple but useful corollary.

Corollary 4. *[8] Let X be a union of graphs in an association scheme with the property that the clique/co-clique bound holds with equality. Assume that C is a maximum clique and S is a maximum independent set in X with characteristic vectors x and y respectively. If E_j are the idempotents of the association scheme, then for $j > 0$ at most one of the vectors $E_j x$ and $E_j y$ is not zero.*

These results will be crucial since it turns out that Γ^M is a union of graphs in an *association scheme*, a unifying object of algebraic combinatorics worthy of its own chapter.

CHAPTER 3

Association Schemes and Coherent Configurations

Most of the association schemes and coherent configurations that we consider are constructed from groups; however, we note that there are plenty of interesting constructions that do not involve groups [30]. There is a variety of terminology for talking about some of these objects, so henceforth, we reserve the term *association scheme* for referring to a commutative association scheme and *coherent configuration* for referring to a non-commutative association scheme.[1] A more thorough discussion of the theory of association schemes and coherent configurations can be found in [30–32] and [33] respectively.

3.1. Preliminaries

Definition 2. *An association scheme* $\mathcal{A} = \{A_0, A_1, \cdots, A_n\}$ *over a set* X *is a collection of* $|X| \times |X|$ *binary matrices (associates) that satisfy the following axioms:*

(1) $A_i^T \in \mathcal{A}$

(2) $A_0 = I$ *where* I *is the identity matrix.*

(3) $\sum_{i=0}^{n} A_i = J$ *where* J *is the all-ones matrix.*

(4) $A_i A_j = \sum_{k=0}^{n} p_{ij}^k A_k = A_j A_i$

The numbers p_{ij}^k are referred to as the *intersection numbers* of $\mathcal{A}$. It turns out that each A_i can be interpreted as a regular graph and the *degree* k_i of A_i is the number of edges incident to a vertex. If for every associate A_i we have $A_i^T = A_i$, then we say that $\mathcal{A}$ is a *symmetric* association scheme. A collection of $|X| \times |X|$ binary matrices $\mathcal{A}$ over a set X is a *coherent configuration* if all the axioms of Definition 2 hold except for axiom 4, that is,

[1]Technically we should use the term *homogeneous coherent configuration*, but this will create no confusion since every coherent configuration in this work is homogeneous.

$A_iA_j = \sum_{k=0}^n p_{ij}^k A_k \neq A_jA_i$. Note that $A_i^T = A_i \ \forall A_i \in \mathcal{A}$ implies that $A_iA_j = A_jA_i$. It is not hard to see that symmetric association schemes can also be seen as colorings of the edges of the complete graph satisfying nice regularity conditions.

Example 4. *Let $\mathcal{A} = A_0, A_1, \cdots, A_n$ be the set permutation matrices of a commutative group G where $A_0 = I$. It is not hard to verify that $\mathcal{A}$ is an association scheme.*

This example demonstrates that association schemes generalize Abelian groups. The following shows that we can in fact construct association schemes from arbitrary groups.

Example 5. *Let $C_0, C_1, \cdots, C_k$ be the conjugacy classes of a group G and let $A_{C_i} = \sum_{c \in C_i} A_c$ where A_c is the permutation matrix of the group element c. Since $A_{C_i}A_{C_j} = A_{C_j}A_{C_i}$, we have that $\{A_{C_i}\}_{0 \leq i \leq n}$ is an association scheme.*

This association scheme is often referred to as the *conjugacy class association scheme*. Recalling that conjugacy classes coincide with group elements for Abelian groups, we see that Example 4 is a special case of Example 5.

The $n+1$ associates of an association scheme $\mathcal{A}$ over X generate a commutative associative subalgebra $\mathfrak{A}$ of $\mathbb{C}^{|X| \times |X|}$ called the *Bose-Mesner algebra* of $\mathcal{A}$. It is well-known that this algebra is semisimple, hence it admits a unique basis of *primitive idempotents* $E_0, E_1, E_2, \cdots, E_n$. Define the ith *multiplicity* $m_i := \mathrm{Tr}(E_i) = rank(E_i)$. The change of base matrices P, Q between these two bases are defined as follows:

$$A_i = \sum_{j=0}^n P_{ji} E_j; \ E_i = \frac{1}{|X|} \sum_{j=0}^n Q_{ji} A_j$$

The matrices P and Q are referred to as the *first* and *second eigenmatrices* respectively. It is always the case that

$$k_i = P_{0i}; \quad m_i = Q_{0i}.$$

and $PQ = QP = |X|I$. Adopting Delsarte's notation, let $p_i(j) = p_{ji}$ and $q_i(j) = q_{ji}$. The values $p_i(j)$ $(0 \leq i, j \leq n)$ give the eigenvalues of $A_i \in \mathcal{A}$ and if $\Gamma = \bigcup_{i \in \Lambda} A_i$ is a union of members of $\mathcal{A}$, then the eigenvalues are additive, that is, the jth eigenvalue of Γ is $\sum_{i \in \Lambda} p_i(j)$.

It is well-known that when $\mathcal{A}$ is the conjugacy class association scheme of G, the entries of P correspond to the character table of G. For this reason, the matrix P often referred to as the *character table of* $\mathcal{A}$, since it can be viewed as a natural generalization of the character table of a finite group. The conjugacy class association scheme has received some attention in algebraic combinatorics as it gives an association scheme-theoretical framework for determining the eigenvalues of any normal Cayley graph. In particular, the permutation derangement graph $\Gamma(S_n, D_n)$ has been analyzed in this framework [8]. Since the representation theory of S_n and the representation theory of the Bose-Mesner algebra $\mathfrak{A}$ of S_n's conjugacy class association scheme $\mathcal{A}$ coincide, the theory of association schemes gives an alternate but equivalent language for expressing representation theory of S_n. It follows that the eigenvalues of $\Gamma(S_n, D_n)$ can be described in terms of the P matrix of $\mathcal{A}$.

This combinatorial language for expressing group representation theory is interesting, but most would agree that the true power of association schemes arises when the P matrix describes the character theory of a semisimple algebra $\mathfrak{A}$ that is not a group algebra. The *Hamming scheme* and *Johnson scheme* are two well-known examples of (symmetric) association schemes whose associates do not correspond to group conjugacy classes and whose character tables aren't group character tables [32].

From Definition 2, it is clear that homogeneous coherent configurations generalize association schemes; however, their theory has proven to be quite difficult. It is not hard to see that the permutation representation of any finite group is a coherent configuration. Indeed, Higman introduced coherent configurations "to do group theory without the group" [33].

In the next section, we describe a general class of (homogeneous) coherent configurations that arise from the action of a finite group G on $G/H \times G/H$, where G/H is the set of left cosets of $H \leq G$. We shall see that there are a handful of exceptional cases where this construction provides us with the nicest possible coherent configuration, that is, a symmetric association scheme. One of these accidents will serve as the framework for our main result.

3.2. Double Coset Coherent Configurations

A more detailed exposition of all the results in this section can be found in the works [31, 33]. The following gives a straightforward way to construct a coherent configuration given an arbitrary $H \leq G$.

Let G be a group, $H, K \leq G$ be subgroups, and $\Omega = G/H$. Define the (H,K)-*double coset* of $g \in G$ by $HgK = \{hgk : h \in H, k \in K\}$. Like cosets of G, it is well known that the (H,K)-double cosets form an equivalence relation over G and hence partition G. Let G act diagonally on $\Omega \times \Omega = G/H \times G/H$, that is, $g(fH, kH) = (gfH, gkH)$. Let $\mathcal{A} = \{A_0, A_1, \cdots, A_n\}$ be the orbits of the diagonal action where $A_0 = \{(\omega, \omega) : \omega \in \Omega\}$. We shall refer to the elements of $\mathcal{A}$ as *orbitals*. For each $\alpha \in \Omega$, let $A_i(\alpha) = \{\beta \in \Omega : (\alpha, \beta) \in A_i\}$. Then $A_0(\alpha), A_1(\alpha), \cdots, A_n(\alpha)$ are the orbits of G_α on Ω where G_α is the stabilzer of $\alpha \in \Omega$ in G. Set $k_i = |A_i(\alpha)|$ which is independent of the choice of α. Since A_i^T is also an orbit of G, there exists some i' for which $A_i^T = A_{i'}$ and $k_i = k_{i'}$. We call $A_{i'}$ the *paired orbital* of A_i.

The orbitals of $\mathcal{A}$ via

$$G(gH, kH) \mapsto Hgk^{-1}H$$

are (canonically) in bijection with the double cosets $H\backslash G/H$ of G.

Now let ϕ be the (left regular) permutation representation of G on Ω where $(\phi(g))_{xy} = 1$ if $gx = y$; otherwise $(\phi(g))_{xy} = 0 \ \forall g \in G, \ x, y \in \Omega$. Let $\mathfrak{U} \leq GL_{|\Omega|}(\mathbb{C})$ be the set of all $|\Omega| \times |\Omega|$ matrices that commute with $\{\phi(g)\}_{g\in G} \leq GL_{|\Omega|}(\mathbb{C})$. Then $\mathfrak{U}$ is the *centralizer algebra* or *Hecke algebra* of ϕ with basis $\{\phi(g)\}_{g\in G}$. It turns out that the Hecke algebra $\mathfrak{U}$ is spanned by the orbitals of $\mathcal{A}$ as a linear space, which implies that $\mathfrak{U}$ is congruent to the (non-commutative) adjacency algebra $\mathfrak{A}$ generated by $\mathcal{A}$. It follows that $\mathcal{A}$ is a coherent configuration and each member of $\mathcal{A}$ corresponds to a (H, H)-double coset of G. In general, we cannot expect $\mathcal{A}$ to be an association scheme, but the following theorem gives a few conditions for when this does hold [31].

THEOREM 17. *Let $\mathcal{A}$ be a double coset coherent configuration. Then the following are equivalent:*

(1) *$\mathcal{A}$ is an association scheme.*

(2) *The Hecke algebra $\mathfrak{A}$ is commutative (in which case $\mathfrak{A}$ is a Bose-Mesner algebra).*

(3) *Every irreducible representation of the permutation representation 1_H^G (G acting on G/H) occurs with multiplicity one.*

Furthermore, $\mathcal{A}$ is a symmetric association scheme if and only if the action of G on G/H is generously transitive.

When $\mathcal{A}$ is an association scheme, then there is a straightforward way to compute the eigenmatrices [31]. Recall that the character table P of $\mathcal{A}$ is a square matrix whose rows are indexed by the irreducibles of 1_H^G and whose columns are indexed by the associate classes.

THEOREM 18. *Let Hg_iH be the double coset in bijection with $A_i \in \mathcal{A}$. Then the character table P of a commutative double coset coherent configuration can be computed as follows.*

$$P_{ji'} = p_{i'}(j) = \frac{1}{|H|} \sum_{x \in Hg_iH} \chi_j(x) = \frac{1}{|H|} \sum_{C_k} |Hg_iH \cap C_k| \chi_j(c_k)$$

Moreover, the Q eigenmatrix can computed from the character table

$$Q_{ji} = q_i(j) = \frac{p_j(i) m_i}{k_j} = \frac{1}{k_j |H|} \sum_{C_k} |Hg_{j'}H \cap C_k| \chi_i(c_k)$$

where the sums range over the conjugacy classes of G, c_k is an arbitrary representative of C_k, and i' is the index of the orbital paired with i in $\mathcal{A}$.

In the case where $\mathcal{A}$ is non-commutative it is no longer the case that the adjacency algebra $\mathfrak{A}$ is commutative, but the permutation representation 1_H^G is still semisimple [33]. This implies that $\mathfrak{A}$ can be decomposed into a direct sum of irreducibles, each occuring with some multiplicity. Higman shows that the multiplicities of irreducibles (as well as most other properties) is completely determined by the intersection numbers of the coherent configuration, which can be computed as follows [33].

THEOREM 19 (Higman 1975). *Let $\mathcal{A}$ be a double coset coherent configuration. Then the intersection numbers p_{ij}^k are given by*

$$p_{ij}^k = \frac{1}{|H|}|Hg_jH \cap g_kHg_i^{-1}H|$$

.

A non-commutative version of Theorem 18 for computing character tables of double coset coherent configurations would be desirable since the above result is a rather indirect and computationally involved route.

Although the aforementioned construction applies to any subgroup $H \leq G$, we will only be concerned with the case where $H \leq S_{kn}$ is a subgroup of the form $S_k \wr S_n$ or $S_n \wr S_k$. In the next section, we introduce the $H_n \backslash S_{2n}/H_n$-association scheme and provide some examples for small n.

3.3. The Matching Association Scheme

The *matching association scheme* $\mathcal{A}^M$ is the $H_n \backslash S_{2n}/H_n$-coherent configuration. Since the action of S_{2n} on M_{2n} is generously transitive, by Theorem 17, $\mathcal{A}^M$ is in fact a symmetric association scheme. The theorem below gives a bijection between $H_n \backslash S_{2n}/H_n$ and $\mathbf{Part}(n)$.

THEOREM 20 ([34] MacDonald). *Let $w, w_1 \in S_{2n}$. Two matchings we, w_1e have the same cycle type $\lambda \vdash n$ if and only if $w_1 \in H_n w H_n$.*

PROOF. Let $\Gamma(w) = \Gamma(e, we)$. Color the edges of $e = \{e_1, e_2, \cdots, e_n\}$ red and color the others blue. The machings we, w_1e have the same cycle type if and only if $\Gamma(w) \cong \Gamma(w_1)$, that is to say if and only if there is a permutation $h \in S_{2n}$ that preserves edge-colors and maps $\Gamma(w)$ onto $\Gamma(w_1)$. Since h sends red edges to red edges, we have $h \in H_n$ and since the blue edges $w_1 e_i$ of $\Gamma(w_1)$ are a permutation of the blue edges hwe_i of $h\Gamma(w)$, the e_i are a permutation of the $w_1^{-1}hwe_i$, whence $w_1^{-1}hw \in H$, and consequently $w_1 \in H_n w H_n$. $\square$

It follows that we may identify each double coset $H_n \backslash \pi / H_n$ with its cycle type $\lambda \vdash n$ and therefore each associate (orbital) $A_\lambda \in \mathcal{A}^M$ admits the following definition.

$$(A_\lambda)_{ij} = \begin{cases} 1, & \text{if } d(i,j) = \lambda \\ 0, & \text{otherwise} \end{cases}$$

Since $\mathcal{A}^M$ is a symmetric $(i = i')$ association scheme, $A_\lambda^T = A_\lambda$ and $A_\lambda A_\mu = A_\mu A_\lambda\ \forall \lambda, \mu \vdash n$. The degree k_λ of the λ-associate is simply the number of matchings whose cycle type is λ. The character table of $\mathcal{A}^M$ can be filled out as follows

$$p_\lambda(\mu) = \frac{1}{|H|} \sum_{\zeta \vdash 2n} |Hg_\lambda H \cap C_\zeta| \chi_\mu^\zeta$$

where $Hg_\lambda H$ is the coset corresponding to cycle type $\lambda \vdash n$ and C_ζ is the conjugacy class of S_{2n} corresponding to ζ. In the next chapter, we give a simplification of the above formula and a discussion of the primitive idempotents $\{E_\lambda\}_{\lambda \vdash n}$ of $\mathcal{A}^M$ in terms of zonal spherical functions of Gelfand pairs.

It turns out that the matching association scheme is a special case of a *meet table* coherent configuration [35] which has been studied extensively by Meagher and is defined as follows. Let $P = \{p_1, p_2, \cdots, p_k\}$ and $Q = \{q_1, q_2, \cdots, q_k\}$ be k-uniform partitions of $[nk]$, that is, partitions of $[nk]$ such that each part p_i has size $|p_i| = n$. Define the $k \times k$ *meet table* $T(P, Q)$ of P and Q (or vice versa) by $(T(P, Q))_{ij} = |p_i \cap q_j|$. Since the ordering of the parts is immaterial, two meet tables are isomorphic if and only if one can be obtained from the other via permutations of the rows and/or columns. Let X be the set of all k-uniform partitions and let d be the number of non-isomorphic meet tables over X. Let $\mathcal{A}^k$ be the set of d non-isomorphic $|X| \times |X|$ binary matrices as follows.

$$(A_T)_{ij} = \begin{cases} 1, & \text{if } T(i,j) \cong T \\ 0, & \text{otherwise} \end{cases}$$

Clearly, these matrices partition $X \times X$. Meagher showed that $\mathcal{A}^k$ is always a coherent configuration and is a symmetric association scheme for $k = 2, \frac{n}{2}$. For $k = \frac{n}{2}$, it turns out that the meet table relation for two matchings is actually isomorphic to our relation used for defining the associates of $\mathcal{A}^M$.

PROPOSITION 1. $\mathcal{A}^M \cong \mathcal{A}^{\frac{n}{2}}$

PROOF. Let $x, y \in M_{2n}$ and let $\mathrm{L}(x,y)$ be the multiline graph of $\Gamma(x,y)$ defined as follows. If $e, e' \in E$ are edges such that $e \neq e'$ as sets, then e is incident to e' in $\mathrm{L}(x,y)$ if and only if e and e' share an endpoint. If $e = e'$ as sets, then that edge is incident to itself twice in $\mathrm{L}(x,y)$.

Since $\Gamma(x,y)$ is 2-regular, it follows that $\mathrm{L}(x,y)$ is a 2-regular graph with loops. The non-isomorphic $n/2 \times n/2$ meet tables of $n/2$-uniform partitions are in one-to-one correspondence with the non-isomorphic 2-regular graphs on $2n$ vertices with loops which in turn are in one-to-one correspondence with the integer partitions of n. This implies that the associate classes of $\mathcal{A}^M$ are isomorphic to those of $\mathcal{A}^{\frac{n}{2}}$, which implies that $\mathcal{A}^M$ is isomorphic to $\mathcal{A}^{\frac{n}{2}}$ as an association scheme. $\square$

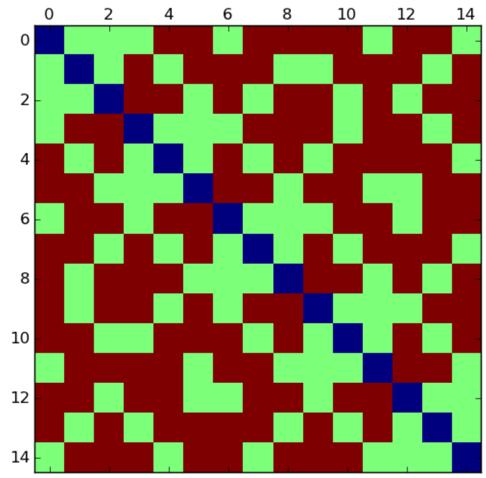

FIGURE 3.1. An edge coloring of K_{15} corresponding to the matching association scheme for $n = 3$. Γ_n^M is the graph induced by the maroon edges.

Since $S_k \wr S_n$ is the stabilizer of n/k-uniform partition of $[nk]$, it is easy to see that meet table coherent configurations are precisely the $(S_k \wr S_n)\backslash S_{kn}/(S_k \wr S_n)$-double coset coherent configurations.

DATA FOR SMALL n. Below we give data for the matching association schemes where $n = 3, 4, 5, 6$. The character tables of these schemes have been computed before [36]. The associates are given as a coloring of the edges of the complete graph. Each of the adjacency matrices of the complete graphs have been sorted with respect to a lexicographic order on the matchings. Each color corresponds to a λ-associate of the matching association scheme $\mathcal{A}$ where $\lambda \vdash n$. The double cosets were computed using `GAP` [37] and the irreducible character calculations were computed using the author's implementation of the Murnaghan-Nakayama rule.[2] We also list the spectra of the matching derangement graphs for $n = 3, 4, 5, 6$.

[2]This code can be found on the author's website.

TABLE 3.1. The character table of $\mathcal{A}_3^M$.

1^3	$2,1$	3
1	6	8
1	1	-2
1	-3	2

TABLE 3.2. A partition of S_{2n} where $n = 3$. The entry (λ, μ) corresponds to the number of permutations σ of cycle-type $\mu \vdash 2n$ such that $d(e, \sigma e) = \lambda$ where $\lambda \vdash n$. Both the columns and the rows are sorted in lexicographical order.

1	3	9	7	0	0	8	6	6	0	8
0	12	12	0	24	72	0	12	60	48	48
0	0	24	8	16	48	32	72	24	96	64

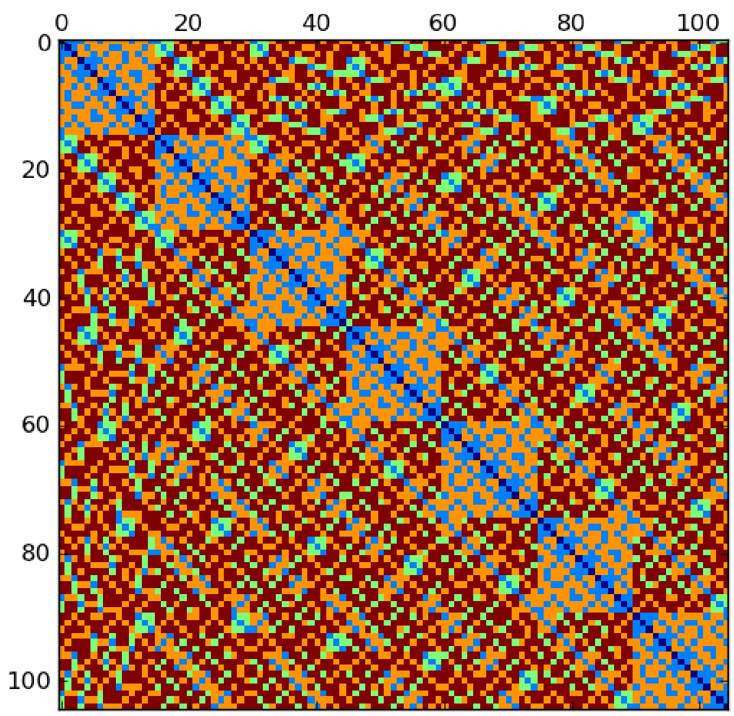

FIGURE 3.2. An edge coloring of K_{105} corresponding to the matching association scheme for $n = 4$. Γ_n^M is the graph induced by the maroon and green edges.

Table 3.3. The character table of $\mathcal{A}_4^M$.

1^5	$2,1^2$	2^2	$3,1$	4
1	12	32	12	48
1	5	4	-2	-8
1	2	-8	7	-2
1	-1	-2	-2	4
1	-6	8	3	-6

Table 3.4. A partition of S_{2n} where $n = 4$. The entry (λ, μ) corresponds to the number of permutations σ of cycle-type $\mu \vdash 2n$ such that $d(e, \sigma e) = \lambda$ where $\lambda \vdash n$. Both the columns and the rows are sorted in lexicographical order. The first column has been omitted.

4	18	28	25	0	0	0	32	32	12	24	36	0	60	0	0	0	32	32	0	48
24	48	72	0	48	288	336	0	0	24	288	456	480	48	192	192	384	192	576	384	576
0	96	128	32	64	256	576	320	704	288	768	96	384	384	384	1920	768	832	1216	1536	1536
0	48	0	48	0	192	0	384	0	0	288	192	576	384	0	384	0	384	384	768	576
0	0	192	0	0	384	768	384	384	96	1152	480	1920	384	768	1536	1536	1920	1152	3072	2304

Table 3.5. The character table of $\mathcal{A}_5^M$

1^5	$2,1^3$	$2^2,1$	$3,1^2$	$3,2$	$4,1$	5
1	20	60	80	160	240	384
1	11	6	26	-20	24	-48
1	6	11	-4	20	-26	-8
1	3	-10	2	-4	-8	16
1	0	5	-10	-10	10	4
1	-4	-3	2	10	6	-12
1	-10	15	20	-20	-30	24

Table 3.6. The character table of $\mathcal{A}_6^M$

1^6	$2,1^4$	$2^2,1^2$	2^3	$3,1^3$	$3,2,1$	$3,3$	$4,1^2$	$4,2$	$5,1$	6
1	30	180	120	160	960	640	120	720	1440	3840
1	19	48	-12	72	80	-64	-12	192	-144	-384
1	12	27	30	16	24	-8	30	-18	108	-48
1	9	-12	-12	22	-60	16	-12	12	-24	96
1	9	33	-27	-8	120	136	-27	-78	-114	-24
1	4	3	-2	-8	0	-24	-2	-18	-4	16
1	0	-21	6	4	12	16	6	-6	12	-48
1	0	15	30	-20	-60	40	30	30	-60	0
1	-3	3	-9	-8	0	4	-9	24	24	-12
1	-8	3	6	12	20	-16	6	-6	-36	48
1	-15	45	-15	40	-120	40	-15	-90	90	-120

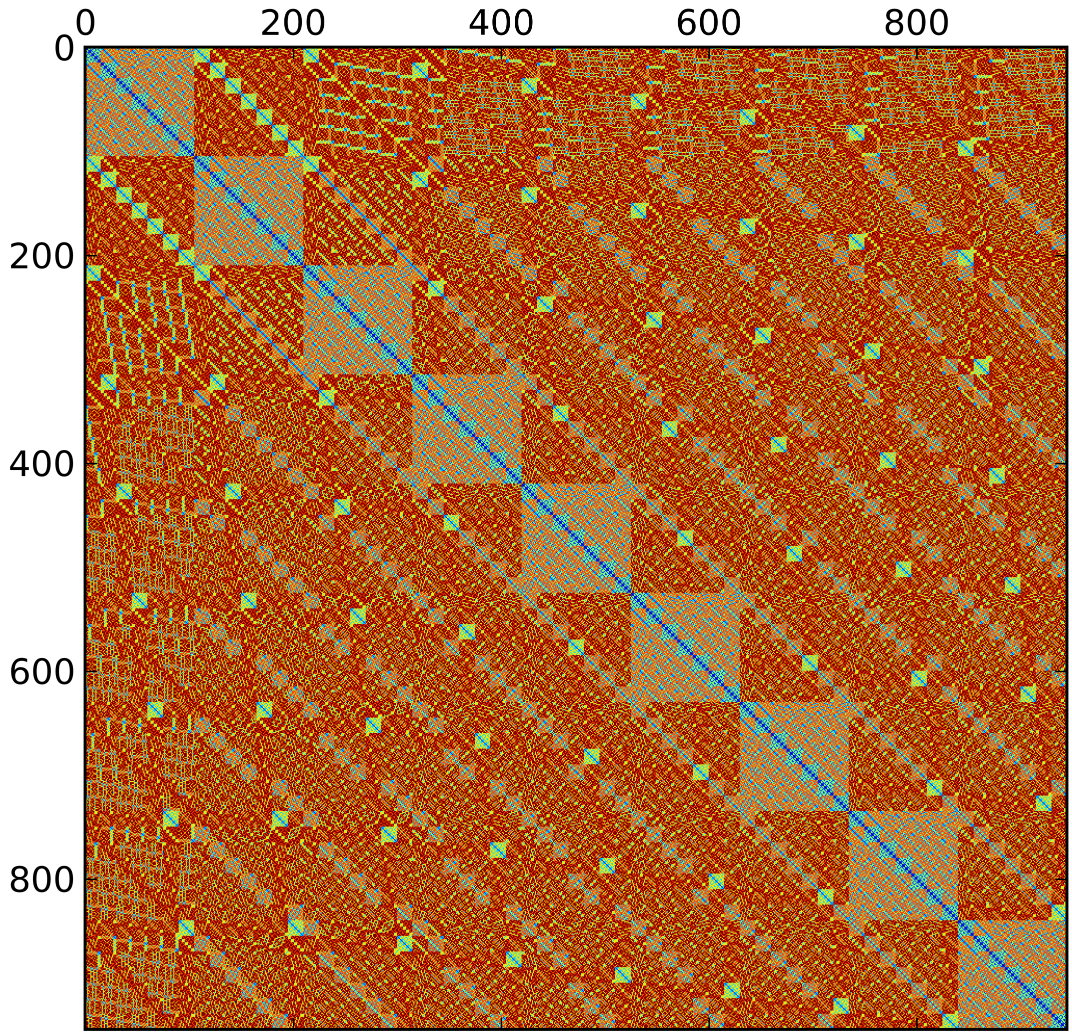

FIGURE 3.3. An edge coloring of K_{945} corresponding to the matching association scheme for $n = 5$. Γ_n^M is the graph induced by the colors that are not present along the 105×105-block diagonal.

TABLE 3.7. The spectra of Γ_n^M for $n = 3, 4, 5, 6$. The multiplicity of the eigenvalue corresponding to $\lambda \vdash n$ is given by Corollary 5.

1^3	$2,1$	3
2	-2	8

1^4	$2,1^2$	$2,2$	$3,1$	4
-3	2	5	-10	60

1^5	$2,1^3$	$2^2,1$	$3,1^2$	$3,2$	$4,1$	5
4	-3	-6	12	12	-68	544

1^6	$2,1^4$	$2^2,1^2$	2^3	$3,1^3$	$3,2,1$	3^2	$4,1^2$	$4,2$	$5,1$	6
-29	2	70	10	-14	-14	-5	76	82	-604	6040

CHAPTER 4

Finite Gelfand Pairs

The theory of Gelfand pairs and zonal spherical functions was originally developed in the context of the representation theory of locally compact groups to study the harmonics of the 2-sphere $SO(3)/SO(2)$ and other generalizations; however, the theory does not degenerate if we require the groups to be finite. It is well-known that the characters of the irreducible representations of a group G form an orthonormal basis of $L^2(\mathbb{C}[G])$ and allows one to conduct harmonic analysis over G. Roughly speaking, the theory below is an extension that tells us when traditional harmonic analysis over cosets G/K is feasible even though $K \ntrianglelefteq G$. The results below can be found in [34, 38].

4.1. Gelfand Pairs and Their Zonal Spherical Functions

Let $L^2(\mathbb{C}[G])$ be the space of complex-valued functions on G. Recall that multiplication corresponds to convolution and the left regular action is given by $(xf)(y) = f(x^{-1}y)$. For any choice of $K \leq G$ there is a corresponding subalgebra $C(G, K) \leq L^2(\mathbb{C}[G])$ of functions that are constant on each double coset KxK in G, more formally, $C(G, K) = \{f : f(kxk') = f(x)\ \forall x \in G,\ \forall k, k' \in K\}$.

Theorem 21. *Let $K \leq G$ be a group. Then the following are equivalent.*

(1) *(G, K) is a Gelfand Pair;*

(2) *The induced representation 1_K^G (permutation representation of G acting on G/K) is multiplicity-free;*

(3) *The algebra $C(G, K)$ is commutative.*

For an arbitrary group G, it is well-known that $(G \times G, G)$ is a Gelfand pair whose decomposition into irreducibles is essentially that of $\mathbb{C}[G]$, hence the theory of Gelfand pairs can be seen as a generalization of group representation theory. It follows that a $K\backslash G/K$-coherent configuration is an association scheme if and only if (G, K) is a Gelfand Pair. Also, we have $C(G, K) \cong \mathfrak{A}$ where $\mathfrak{A}$ is the Hecke (adjacency) algebra of the coherent configuration. The lemma below gives a computationally friendly way of checking that certain $K \leq G$ form a Gelfand pair.

Lemma 3. *If $KgK = Kg^{-1}K$ $\forall g \in G$, then (G, K) is a Gelfand Pair.*

Now let $x_0, x_1, \cdots, x_n \in G$ be distinct representatives of the double cosets $K\backslash G/K$. Then for each double coset Kx_iK, define the ith *sphere* as follows.

$$\Omega_i := \{xK : x \in Kx_iK\}$$

It is clear that $\{\Omega_i\}_{0 \leq i \leq n}$ partitions the left cosets G/K and the degrees of the $K\backslash G/K$-coherent configuration correspond to sphere sizes, that is, $k_i = |\Omega_i|$. In short time we will see that spheres essentially serve the role of conjugacy classes. For the remainder of the section, assume that (G, K) is a Gelfand pair. Define the *zonal spherical function* of (G, K) corresponding to the irreducible i as follows:

$$\omega_i(x) = \frac{1}{|K|} \sum_{k \in K} \chi_i(x^{-1}k)$$

If χ_i is real-valued, then we have $\omega_i = \frac{1}{|K|} \sum_{k \in K} \chi_i(xk)$, which we shall later prefer in light of the fact that the characters of the symmetric group are integral. It follows that $\omega_i(kx) = \omega_i(xk) = \omega(x)$ $\forall x \in G$, $\forall k \in K$ and like the irreducible character functions, the zonal spherical functions $\{\omega_i\}$ form an orthogonal basis of $C(G,K)$ [34]. Recall that the irreducible character functions are constant on conjugacy classes, so we may write χ_i^j for the value of x_i over the conjugacy class C_j. Since the zonal spherical functions are constant on the $K\backslash G/K$ double cosets, we may write ω_i^j where j is the index of a double coset Kx_jK. Essentially, the zonal spherical functions ω_i^j of $C(G,K)$ serve the same role as the irreducible character functions χ_i^j of $\mathbb{C}[G]$. It is known that the character table of a double coset coherent configuration is determined by its zonal spherical functions [31].

THEOREM 22. *Let $\Gamma = \bigcup_j^{\Lambda} A_j$ be a union of graphs in a $K\backslash G/K$-association scheme where (G,K) is a Gelfand pair and Λ is the index set of some subset of the associates. The eigenvalue η_i of Γ corresponding to irreducible i in the multiplicity-free decomposition of 1_K^G can be written as:*

$$\eta_i = \sum_{j\in\Lambda} |\Omega_j|\omega_i^j$$

where ω_i^j is the value of the zonal spherical function corresponding to irreducible i on double coset j. Moreover, η_i occurs with multiplicity $\omega_i(1)$.

It is clear that we obtain Theorem 14 as a corollary if the Gelfand pair is $(G\times G, G)$.

4.2. THE GELFAND PAIR (S_{2n}, H_n)

THEOREM 23. *(S_{2n}, H_n) is a Gelfand Pair*

PROOF. From Theorem 20 it is easy to see that $x^{-1}\in H_nxH_n$ if and only if $x\in H_nx^{-1}H_n$, that is, the matchings ex, ex^{-1} have the same cycle type. Applying Lemma 3 gives the result. $\square$

Now would be a good time for us to point out how lucky we are that (S_{2n}, H_n) is a Gelfand pair. The theorem below is due to Saxl.

THEOREM 24. *Let $n > 18$ and $H \leq S_n$. If $1_H^{S_n}$ is multiplicity free, then one of the following holds:*

(1) *$A_{n-k} \times A_k \leq H \leq S_{n-k} \times S_k$ for some k with $0 \leq k < n/2$;*

(2) *$n = 2k$ and $A_k \times A_k < H \leq S_k \wr S_2$;*

(3) *$n = 2k$ and $H \leq S_2 \wr S_k$ with $[S_2 \wr S_k : H] \leq 4$;*

(4) *$n = 2k+1$ and H xes a point of $[1, n]$ and is one of the subgroups in (2) or (3) on the rest of $[1, n]$;*

(5) *$A_{n-k} \times G_k \leq H \leq S_{n-k} \times G_k$ where $k = 5, 6$ or 9 and G_k is $AGL(1,5), PGL(2,5)$ or $P\Gamma L(2,8)$ respectively.*

This result has been further refined by Godsil and Meagher, who in [39] provide a complete list of the multiplicity-free permutation representations of the symmetric group. By Theorem 24 this gives a complete list of the finite Gelfand pairs of the form (S_n, K) where $K \leq S_n$. The theorem above implies that $H = H_n = S_2 \wr S_n$ and $H = S_n \wr S_2$ are the only two wreath product subgroups of S_{2n} that yield infinite families of multiplicity-free permutation representations. This proves that for $H \neq H_n$ or $(S_n \wr S_2)$, the $H \backslash S_n / H$-coherent configuration is non-commutative.

Since (S_{2n}, H_n) is a Gelfand pair, it follows that the permutation representation S_{2n} on M_{2n} is multiplicity-free. Define $2\lambda := (2\lambda_1, 2\lambda_2, \cdots, 2\lambda_k)$ where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ is an integer partition of n.

Theorem 25 (The M_{2n} Decomposition Theorem). *Let $S^{2\lambda}$ be the irreducible Specht module corresponding to the partition $2\lambda \vdash 2n$. Then*

$$1_{H_n}^{S_{2n}} = \bigoplus_{\lambda \vdash n} S^{2\lambda}$$

The spheres of (S_{2n}, H_n) can be written as follows.

$$\Omega_\lambda = \{m \in M_{2n} : d(e, m) = \lambda\}$$

The spheres partition M_{2n} into classes based on their cycle type much like conjugacy classes do for the symmetric group.[1] Turning back to the matching association scheme $\mathcal{A}^M$, we clearly have $k_\lambda = |\Omega_\lambda|$, but in light of the above decomposition, the multiplicities of $\mathcal{A}^M$ can be computed using the hook length formula, $m_\lambda = \chi_{2\lambda}(1) = h(2\lambda)$.

We can even determine size of each double coset. Let H_ρ be the double coset corresponding to $\rho \vdash n$ and let $l(\rho)$ be the number of parts of ρ. We have the following count [34].

Lemma 4.

$$|H_\rho| = \frac{|H_n|^2}{z_{2\rho}} = \frac{|H_n|^2}{2^{l(\rho)} z_\rho}$$

where z_ρ is the product $\prod_{i \geq 1} i^{m_i} m_i!$ and m_i is the number of parts of ρ equal to i.

The zonal spherical functions can be computed explicitly computed as follows.

$$\omega_\lambda(x) = \frac{1}{|H_n|} \sum_{h \in H_n} \chi_{2\lambda}(xh)$$

[1] To be a bit more precise, we should say that these are the spheres of (S_{2n}, H_n) *centered at e*, the identity matching.

Since zonal spherical functions are constant on double cosets $H_n \backslash S_{2n}/H_n$, we have $\omega^\lambda(x) = \omega^\lambda(y)$ for any $y \in H_n x H_n$, so write ω_λ^μ where $\mu \vdash n$.[2] It turns out that the values ω_λ^μ are essentially the coefficients of the *zonal (symmetric) polynomials* Z_λ [34]. To our knowledge, there is not a simple Murnaghan-Nakayama-like rule for reading off the ω_λ^μ coefficients of zonal polynomials; however, in an unpublished paper, Diaconis and Lander have determined these values for a few $\lambda, \mu \vdash n$ [34]. These formulas will be crucial for obtaining our result.

Lemma 5. $p_{(1^n)}(\lambda) = \omega_\lambda^{(1^n)} = 1$

Lemma 6. *Let $\lambda \vdash n$ be a shape and c be a cell of λ. Let $W(c)$ count the number of cells in c's row that lie west of c and $N(c)$ count the number of cells in c's column that lie north of c. Then*

$$p_{(n)}(\lambda) = \omega_\lambda^{(n)} = \frac{1}{|H_{n-1}|} \prod_{c \in \lambda} (2W(c) - N(c))$$

where the product excludes the cell in the upper-left corner. Moreover, if λ covers 2^3, then $\omega_\lambda^{(n)} = 0$.

Corollary 5. *Let $\Gamma = \bigcup_{\mu \in \Lambda} A_\mu$ be a graph that is the union of some graphs in the matching association scheme. The eigenvalue corresponding to $\lambda \vdash n$ of Γ can be written as:*

$$\eta_\lambda = \sum_{\mu \in \Lambda} |\Omega_\mu| \omega_\lambda^\mu$$

Moreover, η_λ occurs with multiplicity $\omega_\lambda(1) = h(2\lambda)$.

[2]To avoid confusion, we note that our λ and μ are inverted with respect to MacDonald's definition.

THEOREM 26. *Let $N = 2n-1$. The primitive idempotents of the matching association scheme $\mathcal{A}^M$ are defined as follows:*

$$E_\lambda = \frac{m_\lambda}{N!!} \sum_{\mu \vdash n} \frac{p_\mu(\lambda)}{k_\mu} A_\mu$$

$$(E_\lambda)_{x,y} = \frac{m_\lambda}{N!!} \frac{p_{d(x,y)}(\lambda)}{k_{d(x,y)}}$$

Moreover, the entry $(E_\lambda)_{x,y} = 0$ if and only if the zonal spherical function $\omega_\lambda^{d(x,y)} = 0$.

We conclude with a list of Gelfand pairs of the form (S_n, K), along with their decomposition and what combinatorics they describe. Recall that S^λ is the irreducible of S_n corresponding to $\lambda \vdash n$.

- Group Representation Theory $\longleftrightarrow (G \times G, G) \longleftrightarrow \bigoplus_{i=1}^k \dim(V_i) V_i$
- Rep. Theory of Permutations $\longleftrightarrow (S_n \times S_n, S_n) \longleftrightarrow \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda$
- Rep. Theory of k-subsets $\longleftrightarrow (S_n, S_{n-k} \times S_k) \longleftrightarrow \bigoplus_{i=1}^k S^{(n+k-i,i)}$
- Rep. Theory of Perfect Matchings $\longleftrightarrow (S_{2n}, S_2 \wr S_n) \longleftrightarrow \bigoplus_{\lambda \vdash n} S^{2\lambda}$
- Rep. Theory of 2-Uniform Partitions $\longleftrightarrow (S_{2n}, S_n \wr S_2) \longleftrightarrow \bigoplus_{i=0}^{\lfloor n/2 \rfloor} S^{(2n-2i,2i)}$

Unfortunately, we cannot include the representation theory of k-uniform partitions in this list. In the final chapter we will close with speculation on how to cope with the unfortunate reality that $(S_{kn}, S_k \wr S_n)$ is not a Gelfand pair for $k > 2$.

CHAPTER 5

The Erdős-Ko-Rado Theorem for Intersecting Families of Perfect Matchings

A family of matchings $\mathcal{F} \subseteq M_{2n}$ is *intersecting* if $m \cap m' \neq \emptyset$ for all $m, m' \in \mathcal{F}$. We say that a family of matchings $\mathcal{F}$ is *trivially intersecting* if it is a family of the following form.

$$\mathcal{F}_{ij} := \{m \in M_{2n} : \{i, j\} \in m\}$$

In this chapter, we give the first algebraic proof of the following theorem.

Theorem 27 ([40] Meagher, Moura 2005). *If $\mathcal{F}$ is an intersecting family of matchings, then*

$$|\mathcal{F}| \leq (2(n-1)-1)!!$$

If equality holds, then $\mathcal{F}$ is isomorphic to a trivially intersecting family $\mathcal{F}_{ij}$.

The statement was first proven combinatorially as a special case of a more general result on intersecting families of k-uniform partitions. Throughout this chapter, let $N = 2n - 1$ and let v_{ij} be the characteristic 0/1 vector of a trivially intersecting family $\mathcal{F}_{ij}$. An outline of our proof using the module method is listed below.

(1) Define a non-intersection graph Γ^M over M_{2n}.

(2) Show that a maximum independent set of Γ^M meets the clique/co-clique bound with equality.

(3) Show that $v_S - \frac{1}{N}\mathbf{1}$ lives in the "standard" module $S^{2(n-1,1)}$.

(4) Show that if v_S is the characteristic vector of a maximum independent set of Γ^M, then $v_S = v_{ij}$ for some $i, j \in [2n]$.

By "standard" module, we mean the module that is analogous to the standard representation of the symmetric group. In our situation, $S^{2(n-1,1)}$ is our "standard" module since we observed that the irrep $\chi_{2(n-1,1)}$ of $1_{H_n}^{S_{2n}}$ has a natural correspondence to the shape $(n-1,1) \vdash n$ which in turn corresponds to the standard representation $\chi_{(n-1,1)}$ of the symmetric group.

It is clear that for any ground set X and suitably chosen intersection relation $\sim$, showing steps 1 - 4 implies the EKR theorem for intersecting families of X. In the next section, we carry out steps 1 and 2 of the module method.

5.1. The Matching Derangement Graph

We begin by proving some basic properties of the matching derangement graph Γ^M. To illustrate the similarity between the matching derangement graph and the permutation derangement graph $\Gamma(S_n, D_n)$, we recall some basic results of $\Gamma(S_n, D_n)$ then prove their analogues for Γ^M.

Theorem 28. *The size of a maximum clique in $\Gamma(S_n, D_n)$ is n.*

Proof. Clearly no clique of $\Gamma(S_n, D_n)$ can have more than n vertices and any Latin square of order n is clique of $\Gamma(S_n, D_n)$. $\square$

Theorem 29. *The size of a maximum clique in Γ_n^M is $2n-1$.*

Proof. Clearly no clique of Γ can have more than $2n-1$ vertices. Baranyai's theorem states that K_{2n} admits a $(2n-1)$-edge-coloring. Each color class corresponds to a matching and no two matchings intersect, hence there is a clique of size $2n-1$ in Γ^M. $\square$

THEOREM 30. *The size of a maximum independent set in* $\Gamma(S_n, D_n)$ *is* $(n-1)!$.

PROOF. The coset of the stabilzer of a point in S_n is a maximal independent set of size $(n-1)!$ in $\Gamma(S_n, D_n)$. Since $\Gamma(S_n, D_n)$ is a normal Cayley graph, it follows that $\Gamma(S_n, D_n)$ is a union of graphs in the conjugacy class association scheme on S_n. Applying the clique.co-clique bound and Theorem 28 gives the result. $\square$

Since Γ^M is regular and S_{2n} acts transitively on M_{2n}, Γ_n^M is vertex-transitive; however, no group acts regularly on M_{2n}, so Γ^M is not a Cayley graph. The absence of a group structure on the vertices forces us to use the more general representation-theoretic results developed in the previous chapters.

PROPOSITION 2. Γ^M *is a union of members of the matching association scheme* $\mathcal{A}^M$.

PROOF. $\Gamma^M = \bigcup_\lambda A_\lambda$ where λ ranges over integer partitions that correspond to cycle types of S_n that contain no 1-cycle. $\square$

THEOREM 31. *The size of a maximum independent in* Γ^M *is* $(2(n-1)-1)!!$.

PROOF. Any independent set that corresponds to $\mathcal{F}_ij$ is a maximal independent set. Applying Proposition 2 along with the clique/co-clique bound and Theorem 29 gives the result. $\square$

THEOREM 32. *The chromatic number of* Γ_n^M *is* $2n-1$.

PROOF. Clearly the chromatic number is greater than or equal to the clique number $2n-1$. Each member of the vertex partition $(\mathcal{F}_{1,2}, \mathcal{F}_{1,3}, \cdots, \mathcal{F}_{1,2n})$ is an independent set which gives rise to a $(2n-1)$-coloring of the vertices of Γ^M. $\square$

Theorem 33. *D_n^M and $-D_n^M/(2n-1)$ are eigenvalues of Γ_n^M.*

Proof. Γ^M admits an equitable partition $(\mathcal{F}_{i,j}, M_{2n} \setminus \mathcal{F}_{i,j})$ with quotient matrix:

$$\begin{pmatrix} 0 & D_n^M \\ \frac{D_n^M}{2n-1} & D_n^M - \frac{D_n^M}{2n-1} \end{pmatrix}$$

whose eigenvalues are D_n^M and $-D_n^M/(2n-1)$. By Lemma 2, these eigenvalues are always eigenvalues of Γ_n^M. $\square$

For $n = 3, 4, 5, 6$, it has been verified in `GAP` that Γ_n^M meets the Delsarte-Hoffman bound with equality which motivates the following conjecture.

Conjecture 1. *The least eigenvalue of the matching derangement graph is $-\frac{D_n^M}{2n-1}$.*

In [12], Ku and Wales show that the spectrum of the permutation derangement graph possesses the so-called *alternating sign property*. The tables of Section 3.3 suggest the following conjecture.

Conjecture 2. *Γ_n^M has the alternating sign property, that is, for any $\lambda \vdash n$*

$$\mathbf{sign}(\eta_{2\lambda}) = (-1)^{|\lambda|-\lambda_1}$$

where $\eta_{2\lambda}$ is an eigenvalue of Γ^M and $|\lambda| - \lambda_1$ is the number of cells under the first row of λ.

In the next section, we carry out the third step of the module method.

5.2. The "Standard" Module

The main result of this section is proof that $v_S - \frac{1}{N}\mathbf{1}$ lives in the "standard" $2(n-1,1)$-module of the multiplicity-free representation $1_{H_n}^{S_{2n}}$ where v_S is the characteristic vector of an arbitrary maximum independent set of Γ^M. Before we delve into this proof, let us take a moment to observe our surroundings.

Our proof thus far fits entirely within the theory of association schemes; however, if we are to the association scheme in a non-trivial manner, then we must get our hands dirty with the representation theory of the Bose-Mesner (adjacency) algebra $\mathfrak{A}$.

At this point in the module method we are truly at the mercy of the adjacency algebra's decomposition into irreducible representations. In the case where $\mathcal{A}$ is a conjugacy class association scheme, the representation theory of $\mathfrak{A}$ is simply group representation theory, but for more exotic (possibly non-commutative) association schemes, the decomposition $\mathfrak{A}$ into irreducible representations is harder to determine.

Recall that if $\mathcal{A}$ is a (non-commutative) double coset $H\backslash G/H$ association scheme (which is a natural case in EKR combinatorics), then its adjacency algebra $\mathfrak{A}$ is isomorphic to the permutation representation 1_H^G. We saw that when $G = H = S_n$, we obtain the left regular representation $1_{S_n}^{S_n}$ whose decomposition contains all of the irreducible representations χ_λ of S_n each occurring with multiplicity $h(\lambda)$. For arbitrary permutation representations 1_H^G, determining which irreducibles appear with what multiplicity in the decomposition is difficult. It is natural to study the circumstances under which this decomposition is as nice as possible, which is one of aims of the theory of Finite Gelfand pairs. Theorem 24 tells us that when $G = S_n$, we do not get a multiplicity-free representation unless we are lucky. Since (S_{2n}, H_n) happens to be a Gelfand pair, the character theory is understood well-enough to calculate the primitive idempotents $\{E_i\}$, which will be of utmost importance in the proof below.

	2	4	6	8	10	12	...
-1	1	3	5	7	9	11	...
-2	**0**						
-3							
-4							
$\vdots$							

FIGURE 5.1. An illustration of the cell values in the product of Lemma 6

We now show that the characteristic vector v_S of any maximum independent set S of Γ_n^M lives in the sum of trivial $2(n)$-module and the "standard" $2(n-1,1)$-module. Our proof is similar to the one given in [8] for permutations.[1]

THEOREM 34. *Let v_S be the characteristic vector of a maximum independent set of Γ_n^M. Then the vector $v_S - \frac{1}{N}\mathbf{1}$ exists in the $2(n-1,1)$-module.*

PROOF. It is immediate that $\mathbf{1}^T(v_S - \frac{1}{N}\mathbf{1}) = 0$, thus the vector cannot exist in the (all-ones) $2(n)$-module. For any maximum clique C of Γ_n^M define

$$\omega_\lambda(C) = \sum_{c \in C} \omega_\lambda(c)$$

Since Γ_n^M meets the clique/co-clique bound with equality, we have Corollary 4 at our disposal. If there exists a maximum clique C such that $\omega_\lambda(C) \neq 0 \ \forall \lambda \neq (n-1,1),(n)$, then by Theorem 26 it follows that $E_\lambda v_C \neq 0 \ \forall \lambda \neq (n-1,1),(n)$, thus $E_\lambda v_S = 0 \ \forall \lambda \neq (n-1,1),(n)$.

A theorem of Lucas illustrated in Figure 2.4 showed that there there is a nice 1-factorization C that includes the identity matching $e \in C$ and for any two matchings $x, y \in C$, $d(x,y) =$

[1]We point out a slight error in their proof. It is not true that $\chi_\lambda^{(n)} = \pm 1$ for all $\lambda \vdash n$. This is only the case if λ is a rim hook (see Corollary 3); otherwise, $\chi_\lambda^{(n)} = 0$. This minor error has no impact on the correctness of their result since if $\chi_\lambda^{(n)} = 0$, then $\chi_\lambda(C) \neq 0$ since it is assumed that $1 \in C$, and clearly $\chi_\lambda(1) \neq 0 \ \forall n > 0$.

(n). By Theorem 29, C is a maximum clique of Γ_n^M. Since zonal spherical functions act like character functions, we may write $\omega_\lambda(C)$ as follows:

$$\omega_\lambda(C) = \sum_{c \in C} \omega_\lambda(c)$$
$$= \omega_\lambda^{1^n} + 2(n-1)\omega_\lambda^{(n)}$$

By Lemma 5, $\omega_\lambda^{1^n} = 1 \; \forall \lambda \vdash n$ and, so it suffices to show such that $\omega_\lambda^{(n)} \neq -\frac{1}{2(n-1)}$ for all $\lambda \neq (n-1,1)$. We prove that $\omega_{(n-1,1)}(C) = 0$, then show that $|\omega_{(n)}^{(n)}| > |\omega_{(n-1,1)}^{(n)}| > |\omega_\lambda^{(n)}|$ for all $\lambda \neq (n), (n-1,1)$.

By Lemma 5 we have $\omega_{(n)}^{(n)} = 1$. By Lemma 6 we have $\omega_{(n-1,1)}(C) = 0$ since $\omega_{(n-1,1)}^{(n)}$ evaluates to $\frac{-|H_{n-2}|}{|H_{n-1}|} = -\frac{1}{2(n-1)}$. We show that $|H_{n-2}|$ is the largest value that the numerator of Lemma 6 can be and it is obtained only when $\lambda = (n-1,1)$.

By Lemma 6 the only $\lambda \vdash n$ that do not evaluate to zero must be of the form $(n-k, 1^k)$ where $0 \leq k < n$ or $(n-j, j-k, 1^k)$ where $0 \leq k < j < n$.

For any shape $\lambda = (n-k, 1^k)$ where $k > \frac{n}{2}$, we have $|\omega_\lambda^{(n)}| < |\omega_{\lambda'}^{(n)}|$ where $\lambda' \vdash n$ is the transpose of λ. It is also easy to see that $|\omega_{(n-k,1^k)}^{(n)}| < |\omega_{(n-k+1,1^{k-1})}^{(n)}|$ where $1 \leq k \leq \frac{n}{2}$, hence $|\omega_{(n-1,1)}^{(n)}| > |\omega_\lambda^{(n)}|$ holds for $\lambda \vdash n$ of the form $(n-k, 1^k)$ where $k > 1$.

Let $\lambda = (n-j, j-k, 1^k)$ where $2 < k < j < n$ and let $\mu = \lambda \setminus \lambda_1$ be shape obtained by removing the first row. For $\mu = (j-k, 1^k)$ where $k > \frac{j}{2}$, using similar reasoning, we have $|\omega_{(\lambda_1,\mu)}^{(j)}| < |\omega_{(\lambda_1,\mu')}^{(j)}|$. It is also true that $|\omega_{(\lambda_1,j-k,1^k)}^{(n)}| < |\omega_{(\lambda_1,j-k+1,1^{k-1})}^{(n)}|$ where $1 \leq k < \frac{j}{2}$. For the case where $1 \leq k \leq 2$, it is easy to see that removing the bottom left cell of λ and placing it in the upper right hand corner always gives a new shape with a larger character sum, hence $|\omega^{(n)}_{(n-1,1)}| > |\omega^{(n)}_{\lambda}|$ for all valid shapes of the form $(n-j, j-k, 1^k)$, which completes the proof. $\square$

Corollary 6. *The minimum eigenvalue of $A_{(n)} \in \mathcal{A}^M$ is $p_{(n-1,1)}((n)) = -|H_{n-2}|$.*

Proof. Since $p_\lambda((n)) = |\Omega_{(n)}|\omega^{(n)}_{\lambda}$ and $p_{(n)}((n)) = |\Omega_{(n)}|\omega^{(n)}_{(n)} = |\Omega_{(n)}|$ is always positive, it follows that $p_{(n-1,1)}((n))$ is the unique least eigenvalue of $A_{(n)}$. Moreover,

$$\begin{aligned} p_{(n-1,1)}((n)) &= |\Omega_{(n)}|\omega^{(n)}_{(n-1,1)} \\ &= |H_{n-1}| - \frac{|H_{n-2}|}{|H_{n-1}|} \qquad \text{(Lemma 4)} \\ &= -|H_{n-2}| \end{aligned}$$

$\square$

The corollary above along with the ratio bound gives the following theorem.

Theorem 35. *Let $\mathcal{F}$ be a family of matchings such that for any two members $x, y \in \mathcal{F}$, $x \cup y$ is disconnected. Then $|\mathcal{F}| \leq (2(n-1)-1)!!$. This bound is tight.*

Proof. By the ratio bound, we have

$$\begin{aligned} |\mathcal{F}| \leq (2n-1)!!\frac{|H_{n-2}|}{|H_{n-1}| + |H_{n-2}|} &= \frac{(2n-1)!}{|H_{n-2}|(2n-1)} \\ &= (2(n-1)-1)!! \end{aligned}$$

Any family of the form $\mathcal{F}_{ij} = \{m \in M_{2n} : \{i,j\} \in m\}$ is maximum independent set of Γ^M_n of size $(2(n-1)-1)!!$ and since $A_{(n)}$ is a subgraph of Γ^M_n, $\mathcal{F}_{ij} = \{m \in M_{2n} : \{i,j\} \in m\}$ must also be a maximum independent set of $A_{(n)}$, hence the bound is tight. $\square$

The result is a bit surprising, since it tells us that a maximum independent set of the matching derangement graph does not increase in size even if we remove all edges except those that belong to the (n)-associate. A similar result has been observed before in the permutation EKR setting [9]. We now carry out the fourth and final step of the module method.

5.3. The Matching Polytope of K_{2n}

We conclude this chapter with a short proof that the trivially intersecting families $\mathcal{F}_{ij}$ are the only intersecting families of size $(2(n-1)-1)!!$. The proof is nearly identical to an unpublished result of Godsil and Meagher's that can be found in [21].

Let M be a $N!! \times \binom{2n}{2}$ matrix whose columns correspond to characteristic vectors v_{ij} of trivially intersecting families of matchings $\mathcal{F}_{ij}$. The rows of M are indexed by the $N!!$ perfect matchings of K_{2n} and each row of M is the characteristic vector of a perfect matching of K_{2n}. It follows that the convex hull of the rows of M is $M(K_{2n})$, the perfect matching polytope of K_{2n} and the rows of M are the vertices of $M(K_{2n})$.

Let z be the characteristic $0/1$ vector of any maximum independent set Z of Γ_n^M. Note that since M has constant row sums, $\mathbf{1}$ is in the column space of M. By Theorem 34, $z - \frac{1}{N}\mathbf{1}$ lives in the "standard" module, hence any z can be expressed $Mh = z$ for some $h \in \mathbb{R}^{\binom{n}{2}}$. Obviously $z_{\min} = 0$ and $z_{\max} = 1$, so by Lemma 1 we have that the rows of M indexed by the support of z correspond to the vertex set of a face F_1 of $M(K_{2n})$ and the remaining rows of M correspond to the vertex set of a face F_0 of $M(K_{2n})$. Moreover, since z is a $0/1$ vector, we have that F_0 and F_1 are parallel faces that partition the vertices of $M(K_{2n})$. Corollary 2 implies that the facets of $M(K_{2n})$ are precisely the sets:

$$F_e = \{x \in M(K_{2n}) : x(e) = 0\}$$

In other words, the vertices of any facet F_e are those matchings of K_{2n} that do not contain $e = \{i, j\} \in E(K_{2n})$ some $i, j \in [2n]$. Since every face lies in a facet, we have that F_0 and F_1 both lie in facets of $M(K_{2n})$. In particular, there is some F_e such that $F_0 \subseteq F_e$, which implies that $x(e) = 1$ for every vertex $x \in F_1$. In other words, z is supported by matchings m such that $e \in m$, of which there $(2(n-1)-1)!!$. Since the support of z can be no larger than $(2(n-1)-1)!!$, it follows that $z = v_{ij}$ for some $i, j \in [2n]$.

CHAPTER 6

Intersecting Families of 1-Factors of Complete Uniform Hypergraphs

In [8], Godsil and Meagher ask whether the module method can be used to prove the EKR theorem for intersecting families of 1-factors (matchings) of the complete r-uniform hypergraph K_r^{rn} for $r > 2$. We have already observed via Theorem 24 that $(S_{rn}, S_r \wr S_{rn})$ is typically not a Gelfand pair, hence the $(S_r \wr S_{rn})\backslash S_{rn}/(S_r \wr S_{rn})$-double coset coherent configuration is not commutative. We conclude with some speculation on how to circumvent this obstacle and a state a few cases where the decomposition of $1_{S_r \wr S_n}^{S_{rn}}$ is known.

6.1. 1-Skeletons of Polytopes

It is probably not accidental that the (n)-associates of both the conjugacy class association scheme of S_n and the matching association scheme are isomorphic to the 1-skeletons of their respective matching polytopes. Note that $\dim S^{(n-1,1)} = \dim M(K_{n,n})$ and $\dim S^{2(n-1,1)} = \dim M(K_{2n})$. Indeed, these polytopes are related to the *eigenpolytopes* [41] of the eigenspace corresponding to the "standard" representation [21]. In [41], Godsil classifies the distance regular regular graphs that are isomorphic to the 1-skeleton of their eigenpolytope corresponding to the second largest eigenvalue. It would be interesting to investigate other association schemes that arise in EKR combinatorics and classify those whose (n)-associate is isomorphic to the 1-skeleton of the eigenpolytope of its "standard" representation.

An obstacle for using the module method for $r > 2$ is that the "standard" representation might occur with multiplicity greater than one; however, it seems plausible that polytopes could be used to infer which representation is the "standard" representation in the decomposition of $1_{S_r \wr S_n}^{S_{rn}}$ along with the multiplicity of the "standard" representation. The polytope $P \subseteq \mathbb{R}^{\binom{rn}{r}}$ whose vertices are the 1-factors of K_r^{rn} is a natural and very symmetric object, so a nice formula for $\dim P$ does not seem too far-fetched.[1]

6.2. Parallelisms

The proof of Theorem 34 relies on the fact that there always exists a "nice" maximum clique in Γ_n^M, that is, a maximum clique whose members all have the same cycle type. Equivalently, for $\binom{[2n]}{2}$, there was a nice parallelism Π where $(n) \vdash n$ was the cycle type of each $\pi \in \Pi$. In light of this, we only had to consider character sums $\chi_\lambda^{(n)}$, which made the character calculations relatively painless. When more cycle types must be considered, then these calculations often become far too complicated. If the module method is to be generalized to K_r^{rn}, then the following questions must be addressed. *What is the "simplest" isomorphism-type for parallelisms of* $\binom{[3n]}{3}$*? Is there a canonical "simplest" isomorphism-type for* $\binom{[kn]}{k}$*?*

6.3. Future Work

From Theorem 24 it follows that $(S_{2n}, S_n \wr S_2)$ is a Gelfand pair, but the combinatorics it describes are the 2-uniform partitions, which are rather uninteresting from an EKR point of view. However, one can interpret $1_{S_n \wr S_2}^{S_{2n}}$ as the multiplicity-free representation theory of n indistinguishable balls in n distinguishable urns, or equivalently, decks of playing cards where one pays attention only to card color. It seems quite possible to obtain Diaconis-type stationarity and mixing results in this arena, which to our knowledge have not been obtained.

[1]A system of linear inequalities that defines the 1-factor polytope of an arbitrary r-uniform hypergraph on rn vertices for $r > 2$ (like Edmonds, Lovasz, and Pulleyblank did for $r = 2$) is probably much too ambitious. There doesn't appear to be an easy way to generalize their result.

Thrall has determined the multiplicities of irreducibles of the decomposition of $1_{S_n \wr S_3}^{S_{3n}}$ [42], which is slightly more interesting than $1_{S_n \wr S_2}^{S_{2n}}$ from an EKR point of view and may be a good place to ease into the non-commutative setting. Finally, it has recently come to the author's attention that Littlewood and Foulkes have determined the multiplicities of irreducibles in the decomposition of $1_{S_k \wr S_n}^{S_{kn}}$ for $k < 7$ [43, 44], so it may be plausible for one to give an algebraic proof of the EKR theorem for intersecting families of 1-factors of the complete r-uniform hypergraph for $r < 7$ via the module method.

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