Technical Report

THE TURBULENT TRANSPORT OF HEAT AND MOMENTUM

by

E. C. Nickerson

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Colorado State University
Fort Collins, Colorado 80521

ABSTRACT

Upper bounds on the heat flux and stress have been obtained for the combined problem of Rayleigh convection and plane Couette flow. An interpretation of the results in terms of bulk eddy exchange coefficients of heat and momentum leads to a value of 1.23 for their ratio in air for near-neutral conditions. The transition between free and forced convection is found to depend upon the Reynolds number as well as the Richardson number.

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1. Introduction

It has been proposed by Malkus (1956), Howard (1963), and Coles (1965), that some classes of turbulent fluids would attempt to reach a state of maximum viscous dissipation. Maximum dissipation corresponds to maximum heat transfer in parallel plate convection, to maximum stress in plane Couette flow, and to maximum torque in cylindrical Couette flow. Since exact solutions to the Navier-Stokes equations are beyond the scope of current mathematical techniques, it is reasonable to seek bounds on the important physical quantities such as momentum flux or heat flux.

Upper bounds on the heat flux, stress, and torque have been obtained by the use of variational methods (see Howard, 1963; Busse, 1969, 1970; Nickerson, 1969) and by the clever use of inequalities (Malkus, 1968). Lindberg (1970) made use of the estimation technique outlined by Malkus (1968) to obtain upper bounds on the heat and salt transfer in thermohaline convection. Howard's (1963) original bounding procedure for disturbances characterized by a single wave number has been extended by Busse (1969, 1970) to include multiwave number disturbances.

The use of upper bounding procedures is a relatively recent development in the study of turbulent flows. Consideration of the combined problem of plane Couette flow and Rayleigh convection, in which the fluxes of heat and momentum are both independent of the vertical coordinate, may lead to results of geophysical interest. References to constant flux layers abound in the meteorological literature (see for example, Lumley and Panofsky, 1964) and the constancy with height of heat, momentum, and water vapor fluxes has proved to be a useful approximation in the lowest few meters of the atmospheric boundary layer.

2. Equations

Let us consider a fluid contained between two plates of infinite horizontal extent. The lower plate, with temperature T_o , moves with a velocity $\frac{\Delta U}{2} \underset{\sim}{i}$. The upper plate, with temperature $T_o - \Delta T$, moves wi h a velocity $-\frac{\Delta U}{2} \underset{\sim}{i}$. The Boussinesq equations that describe the flow field are

$$\rho_o \frac{\partial \overset{*}{\underset{\sim}{V}}}{\partial t} + \rho_o \overset{*}{\underset{\sim}{V}} \cdot \nabla \overset{*}{\underset{\sim}{V}} + \nabla \overset{*}{p} - \rho g \underset{\sim}{k} = \rho_o \nu \nabla^2 \overset{*}{\underset{\sim}{V}} \tag{1}$$

$$\nabla \cdot \overset{*}{\underset{\sim}{V}} = 0 \tag{2}$$

$$\frac{\partial \overset{*}{T}}{\partial t} + \overset{*}{\underset{\sim}{V}} \cdot \nabla \overset{*}{T} = \kappa \nabla^2 \overset{*}{T} \tag{3}$$

$$\rho = \rho_o (1 - \alpha(\overset{*}{T} - T_o)) \quad , \tag{4}$$

where $\underset{\sim}{k}$ is a unit vector in the positive z direction, and ν and κ are the kinematic viscosity and conductivity, respectively.

It will be convenient to separate the velocity vector and temperature into an average over horizontal planes, and departure from this average.

$$\overset{*}{\underset{\sim}{V}} = \overline{\overset{*}{U}}(z)\, \underset{\sim}{i} + \overset{*}{\underset{\sim}{v}} \tag{5}$$

$$\overset{*}{T} = \overline{\overset{*}{T}}(z) + \overset{*}{\theta} \tag{6}$$

At this stage of the analysis it will be useful to make an ergodic assumption concerning the velocity and temperature fields. The time average of any physical quantity at any fixed location within the fluid will be assumed to be identical to an average over a horizontal plane. Therefore, the time rate of change of such a horizontal average is zero.

Upon constructing horizontal averages of (1) and (3), the following relations may be obtained between the mean and fluctuating quantities.

$$\frac{d}{dz}\left(\overline{\overset{*}{w}\overset{*}{u}} - \nu \frac{d\overline{\overset{*}{U}}}{dz}\right) = 0 \tag{7}$$

$$\frac{d}{dz}\left(\overline{\overset{*}{w}\overset{*}{\theta}} - \kappa \frac{d\overline{\overset{*}{T}}}{dz}\right) = 0 \tag{8}$$

where $\overline{(\ )} = \frac{1}{A} \int_A (\ )dA$.

Equations (7) and (8) may be integrated immediately to give the relations

$$\overline{\overset{*}{w}\overset{*}{u}} - \nu \frac{d\overline{\overset{*}{U}}}{dz} = S \tag{9}$$

$$\overline{\overset{*}{w}\overset{*}{\theta}} - \kappa \frac{d\overline{\overset{*}{T}}}{dz} = H \quad . \tag{10}$$

The constant, S and H , on the right side of (9) and (10), represent the horizontally averaged stress and heat flux. Equations (9) and (10) show that for turbulent flow as well as laminar flow, the heat flux and stress are independent of the vertical coordinate.

Equations (9) and (10) may then be integrated in the vertical to obtain an expression for the stress and heat flux as the sum of the laminar and conduction terms, and the values due to the turbulent transfer of momentum and heat within the fluid.

$$\overset{*}{w}\overset{*}{u} + \nu \frac{\Delta U}{D} = S \tag{11}$$

$$\langle \overset{*}{w}\overset{*}{\theta} \rangle + \kappa \frac{\Delta T}{D} = H \quad , \tag{12}$$

where $\langle\rangle = \frac{1}{D} \int_0^D (\ )dz$.

Dimensionless variables are now introduced as

$$\underset{\sim}{x} = \frac{1}{D} \overset{*}{\underset{\sim}{x}} \tag{13}$$

$$\underset{\sim}{v} = \frac{D}{\kappa} \overset{*}{\underset{\sim}{v}} \tag{14}$$

$$\overline{U} = \frac{1}{\Delta U} \overline{\overset{*}{U}} \tag{15}$$

$$T = \frac{1}{\Delta T} \overset{*}{T} \tag{16}$$

$$\tau = \frac{S}{\nu \frac{\Delta U}{D}} \tag{17}$$

$$N = \frac{H}{\kappa \frac{\Delta T}{D}} \quad . \tag{18}$$

The following power integrals may now be constructed from (1) and (3).

$$R < wu \frac{d\overline{U}}{dz} > - Ra<w\theta> = - <|\nabla \underset{\sim}{v}|^2> \tag{19}$$

$$< w\theta \frac{d\overline{T}}{dz} > = - <|\nabla\theta|^2> \quad , \tag{20}$$

where R is the Reynolds number and Ra the Rayleigh number defined by

$$R = \frac{D \ \Delta U}{\nu} \tag{21}$$

$$Ra = \frac{\alpha g \Delta T D^3}{\nu\kappa} \quad . \tag{22}$$

After non-dimensionalizing (9) to (12) according to (13) to (18), one obtains

$$\tau = 1 + P_r^{-2} R^{-1} <wu> \tag{23}$$

$$N = 1 + <w\theta> \tag{24}$$

$$-\frac{d\overline{U}}{dz} = \tau - P_r^{-2}\, R^{-1}\, \overline{wu} = 1 + P_r^{-2}\, R^{-1}[\langle wu\rangle - \overline{wu}] \tag{25}$$

$$-\frac{d\overline{T}}{dz} = N - \overline{w\theta} = 1 + \langle w\theta\rangle - \overline{w\theta} \quad . \tag{26}$$

The substitution of (25) and (26) into (19) and (20) then yields

$$R\langle wu\rangle + P_r^{-2}\, [\langle wu\rangle^2 - \langle \overline{wu}^2\rangle] + Ra\langle w\theta\rangle = \langle|\nabla \underset{\sim}{v}|^2\rangle \tag{27}$$

$$\langle w\theta\rangle + \langle w\theta\rangle^2 - \langle \overline{w\theta}^2\rangle = \langle|\nabla\theta|^2\rangle \quad . \tag{28}$$

The temperature and velocity fluctuations will now be renormalized and new variables defined by

$$\hat{\theta} = \langle wu\rangle^{\frac{1}{2}} \langle w\theta\rangle^{-1}\, \theta \tag{29}$$

$$\underset{\sim}{\hat{v}} = \langle wu\rangle^{\frac{1}{2}}\, \underset{\sim}{v} \quad . \tag{30}$$

The new variables then satisfy the relations

$$\langle\hat{w}\hat{\theta}\rangle = \langle\hat{w}\hat{u}\rangle = 1 \quad . \tag{31}$$

Let us now make the following definitions.

$$\Phi = \langle|\nabla\underset{\sim}{\hat{v}}|^2\rangle \tag{32}$$

$$\Psi = \langle|\nabla\hat{\theta}|^2\rangle \tag{33}$$

$$\Omega = \langle(1 - \hat{w}\hat{\theta})^2\rangle \tag{34}$$

$$\Lambda = \langle(1 - \hat{w}\hat{u})^2\rangle \quad . \tag{35}$$

The substitution of (23), (24), (32) to (35) into (27) and (28) yields

$$\tau - 1 = \Lambda^{-1}[1 - \Phi R^{-1} + Ra\; P_r^{-2}\; R^{-2}(\tau-1)^{-1}(N-1)] \tag{36}$$

$$N-1 = [\Omega + P_r^{-2} R^{-1}(\tau-1)^{-1}\Psi]^{-1} \quad . \tag{37}$$

3. Upper Bounds

In order to obtain upper bounds on N and τ, one must first obtain a lower bound on the integral quantities Ω and Λ.

We first note that for $0 \leq z \leq \frac{1}{2}$,

$$|u(z)| = |\int_0^z u'(z_1)dz_1| \leq [\int_0^z (u') \ dz_1]^{\frac{1}{2}} \ [\int_0^z dz_1]^{\frac{1}{2}} \quad . \tag{38}$$

However, since $\int_0^z (u')^2 dz_1 = \frac{1}{2} \langle u'^2 \rangle - \int_z^{\frac{1}{2}} (u')^2 dz_1$,

we obtain $|u(z)| \leq \frac{1}{\sqrt{2}} \langle (u')^2 \rangle^{\frac{1}{2}} \ z^{\frac{1}{2}}$. (39)

For $\frac{1}{2} \leq z \leq 1$

$$|u(z)| = |\int_z^1 u'(z_1)dz_1| \leq [\int_z^1 (u')^2 dz_1]^{\frac{1}{2}} \ [\int_z^1 dz_1]^{\frac{1}{2}} \quad . \tag{40}$$

However, making use of the identity

$$\int_z^1 (u')^2 dz_1 = \frac{1}{2} \langle u'^2 \rangle - \int_{\frac{1}{2}}^z (u')^2 dz_1 \quad , \tag{41}$$

we obtain

$$|u(z)| \leq \frac{1}{2} \langle (u')^2 \rangle^{\frac{1}{2}} \ (1-z)^{\frac{1}{2}} \quad . \tag{42}$$

Now for $0 \leq z \leq \frac{1}{2}$,

$$|w(z)| = |\int_0^z (z-z_1)w''(z_1)dz_1| \leq [\int_0^z (w'') \ dz_1]^{\frac{1}{2}} \ [\int_0^z (z-z_1)dz_1]^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} \langle (w'')^2 \rangle^{\frac{1}{2}} \ (\frac{z^3}{3})^{\frac{1}{2}} \quad . \tag{43}$$

For $\frac{1}{2} \leq z \leq 1$,

$$|w(z)| = |\int_z^1 (z-z_1)w''(z_1)dz| \leq [\int_z^1 (w'')^2 dz_1]^{\frac{1}{2}} [\int_z^1 (z-z_1)^2 dz]^{\frac{1}{2}} \quad . \quad (44)$$

However, since

$$\int_z^1 (w'')^2 dz_1 = \frac{1}{2} <(w'')^2> - \int_{\frac{1}{2}}^z (w'')^2 dz_1 \quad , \quad (45)$$

we obtain

$$|w(z)| \leq \frac{1}{\sqrt{2}} <(w'')^2>^{\frac{1}{2}} \left(\frac{(1-z)^3}{3}\right)^{\frac{1}{2}} \quad . \quad (46)$$

We then have

$$|wu| \leq \frac{z^2}{z_u^{\,2}} \qquad 0 \leq z \leq \frac{1}{2} \quad (47)$$

$$|wu| \leq \frac{(1-z)^2}{z_u^{\,2}} \qquad \frac{1}{2} \leq z \leq 1 \quad . \quad (48)$$

Similarly,

$$|w\theta| \leq \frac{z^2}{z_\theta^{\,2}} \qquad 0 \leq z \leq \frac{1}{2} \quad (49)$$

$$|w\theta| \leq \frac{(1-z)^2}{z_\theta^{\,2}} \qquad \frac{1}{2} \leq z \leq 1 \quad , \quad (50)$$

where

$$z_u = 2^{\frac{1}{2}} 3^{\frac{1}{4}} <(u')^2>^{-\frac{1}{4}} <(w'')^2>^{-\frac{1}{4}} \quad (51)$$

$$z_\theta = 2^{\frac{1}{2}} 3^{\frac{1}{4}} <(\theta')^2>^{-\frac{1}{4}} <(w'')^2>^{-\frac{1}{4}} \quad . \quad (52)$$

Lower bounds on the integral quantities Ω and Λ in terms of z_u and z_θ may then be obtained from (47) to (52).

$$\Lambda \geq \int_0^{z_u} \left(1 - \frac{z^2}{z_u^2}\right)^2 dz + \int_{1-z_u}^{1} \left(1 - \frac{(1-z)^2}{z_u^2}\right)^2 dz \geq \frac{16}{15} z_u \quad . \tag{53}$$

Similarly,

$$\Omega \geq \frac{16}{15} z_\theta \quad . \tag{54}$$

It is now necessary to express z_u in terms of Φ , and z_θ in terms of Φ and Ψ . This estimation technique is attributed to L.N. Howard by Malkus (1968).

We first note that

$$\Phi \geq \langle (u')^2 + \alpha^2 u^2 \rangle + \langle (w')^2 + \alpha^2 w^2 \rangle \tag{55}$$

and

$$\Phi \geq \langle \frac{1}{\alpha^2} (w'')^2 + 2(w')^2 + \alpha^2 w^2 \rangle \quad , \tag{56}$$

where α is the single horizontal wave number of the separated solution. Since $\langle w^2 \rangle \langle u^2 \rangle \geq \langle wu \rangle^2 = 1$,

$$\langle (u')^2 \rangle \langle (w'')^2 \rangle \leq \alpha^2 [\Phi - \frac{\alpha^2}{\langle u^2 \rangle}] \; [\Phi - \alpha^2 \langle u^2 \rangle]$$

$$= \alpha^2 [\Phi^2 - \alpha^2 \Phi (\frac{1}{\langle u^2 \rangle} + \langle u^2 \rangle) + \alpha^4] \quad . \tag{57}$$

Now since $\dfrac{1}{\langle u^2 \rangle} + \langle u^2 \rangle \geq 2$,

$$\langle (u')^2 \rangle \langle (w'')^2 \rangle \leq \alpha^2 (\Phi - \alpha^2)^2 \quad . \tag{58}$$

The maximum value of the right side of (58) occurs at $\alpha^2 = \Phi/3$, which leads to

$$<(u')^2> \; <(w'')^2> \; \leq \; \frac{4}{27} \; \Phi^3 \quad . \tag{59}$$

After making use of (51), we obtain

$$z_u \geq 3\Phi^{-3/4} \quad . \tag{60}$$

Similarly, one may obtain the relation

$$<(\theta')^2> \; <(w'')^2> \; \leq \; \alpha^2[\Psi - \alpha^2<\theta^2>] \; [\Phi - \frac{\alpha^2}{<\theta^2>}] \quad . \tag{61}$$

Let us now substitute the relations

$$c^2 = \alpha^2<\theta^2>$$

$$b^2 = \frac{\alpha^2}{<\theta^2>}$$

into (61), and find expressions for c and b which will maximize the right side of (61).

$$<(\theta')^2> \; <(w'')^2> \; \leq \; cb[\Psi\Phi - c^2\Phi - b^2\Psi + c^2b^2] \quad . \tag{62}$$

If we differentiate the right side of (62) with respect to c , and set that equal to zero, and also differentiate the right side of (62) with respect to b, and set that equal to zero, we find that maximizing expressions are given by

$$c^2 = \Psi/3 \tag{63}$$

$$b^2 = \Phi/3 \quad . \tag{64}$$

Equation (62) then becomes

$$<(\theta')^2> \; <(w'')^2> \; \leq \; \frac{4}{27} \; \Psi^{3/2} \; \Phi^{3/2} \quad . \tag{65}$$

The final lower bounds on Λ and Ω are then

$$\Lambda \geq \frac{16}{5} \Phi^{-3/4} \tag{66}$$

$$\Omega \geq \frac{16}{5} \Phi^{-3/4} \Psi^{-3/8} \quad , \tag{67}$$

which, together with (36) and (37) yields

$$\tau - 1 \leq \frac{5}{16} \Phi^{3/4} \left[1 - \Phi R^{-1} + \mathrm{Ra}\, P_r^{-2} \left[R \Psi P_r^{-2} + \frac{16}{5} (\tau - 1) \Phi^{-3/8} \Psi^{-3/8}\right]^{-1}\right] \tag{68}$$

$$N - 1 \leq \left[\frac{16}{5} \Phi^{-3/8} \Psi^{-3/8} + P_r^{-2} \Psi R^{-1} (\tau - 1)^{-1}\right]^{-1} \quad . \tag{69}$$

Upper bounds on the stress and heat flux may then be obtained by finding maximizing expressions for Φ and Ψ in terms of the Reynolds, Rayleigh, and Prandtl numbers. The maximizing functions are given by

$$\Phi = \left[\frac{30}{P_r^2 (3R + 8\Phi)}\right]^{8/3} \left[\frac{135\ \mathrm{Ra}}{121(7\Phi - 3R)}\right]^{11/3} \tag{70}$$

$$\Psi = \frac{135\ \mathrm{Ra}}{121(7\Phi - 3R)} \quad . \tag{71}$$

The corresponding bounds on the heat flux and stress in terms of the maximizing functions, are then

$$\tau - 1 \leq \frac{1}{36} \Phi^{3/4} (3 + 8\Phi R^{-1}) \tag{72}$$

$$N - 1 \leq \frac{5}{16} \left[\Phi^{-3/8} \Psi^{-3/8} + 45\, P_r^{-2} \Psi \Phi^{-3/4} (12R + 32\Phi)^{-1}\right]^{-1} \quad . \tag{73}$$

The upper bounds on the heat flux and stress depend implicitly upon the external parameters, and it is therefore of interest to examine two limiting cases of special interest. The first case to be considered occurs when shear effects dominate, and the second occurs when thermal effects dominate.

In the first case, $\Phi = 3R/7$, and

$$\tau - 1 \leq \frac{4}{7} \left(\frac{3}{7}\right)^{3/4} \frac{5}{16} R^{3/4} , \tag{74}$$

which is identical to that given by Malkus (1968) except that the factor $5\sqrt{2}/16$ obtained by Malkus is here replaced by $5/16$.

The domination of the flow by thermal effect occurs when $\Phi >> R$ and leads to the following asymptotic upper bound on the Nusselt number:

$$N - 1 \leq \frac{8}{11} \left(\frac{5}{16}\right) \left[\left(\frac{3}{11}\right) \left(\frac{45}{77}\right)\right]^{3/8} Ra^{3/8} \approx .11 \, Ra^{3/8} . \tag{75}$$

In the derivation of (73) and (75), an upper bound on the heat flux was obtained with the requirement that the stress also be a maximum. It is possible, however, to obtain an upper bound on the heat flux without requiring that the stress be a maximum. If one inserts the relation

$$\tau - 1 = \Lambda^{-1}[1 - \Phi R^{-1} + Ra[1 - \Omega(N-1)]R^{-1} \Psi^{-1}] \tag{76}$$

into (36), a relation for the Nusselt number is obtained in terms of Φ , Ψ , and the external parameters R , Ra , and Pr. Maximizing expressions for Φ and Ψ when $\Phi >> R$ lead to the expression.

$$N - 1 \leq \frac{8}{11} \left(\frac{5}{16}\right) \left(\frac{3}{11}\right)^{3/8} Ra^{3/8} \approx .14 \, Ra^{3/8} . \tag{77}$$

The coefficient in (77) is identical to the corresponding limiting case considered by Lindberg (1970), although the internal details of the two derivations, such as the normalizations and dependence of Nusselt number on the maximizing function Φ are not the same.

4. Discussion of the Results

Upper bounds on the heat flux and stress for air (Prandtl number = .7) are shown in Figures 1 to 3. For the range of parameters considered (i.e., $10^4 \leq R \leq 10^6$, $10^4 \leq Ra \leq 10^9$) these figures indicate that the Reynolds number effect is dominant. However, at sufficiently large Rayleigh numbers, Figure 3 indicates the existence of a Reynolds number range where the stress decreases with increasing Reynolds number.

For sufficiently high Reynolds numbers, or for sufficiently low Rayleigh numbers, the upper bounds on the stress and heat flux, shown in Figure 1, differ by a constant amount on the log-log plot. For Φ slightly larger than $3R/7$, the upper bound on the heat flux, as given by (73), is also proportional to the three quarter power of the Reynolds number and independent of the Rayleigh number. The ratio of (73) to (72) is

$$\frac{N-1}{\tau-1} = \left(\frac{14}{11}\right)\left(\frac{P_r^2}{2}\right)^{3/11} . \tag{78}$$

If one defines eddy exchange coefficients of heat and momentum as average values over a layer, the ratio of those coefficients in terms of the upper bounds given by (72) and (73) becomes

$$\frac{K_H}{K_M} = \frac{(N-1)}{P_r(\tau-1)} = \left(\frac{14}{11}\right)\left(\frac{1}{2}\right)^{3/11} P_r^{-5/11} . \tag{79}$$

For a Prandtl number of .7, (79) results in a value of Kh/Km approximately equal to 1.23. Typical values of Kh/Km reported on in the literature are of order unity, but for near neutral conditions, which corresponds to the asymptotic limit under consideration, Lumley and Panofsky (1964) indicate that a value of 1.3 is consistent with most observations.

Such close agreement between observational results and the results of this paper may be fortuitous. On the other hand, a model of the atmospheric boundary layer for near neutral conditions, in which heat and momentum are transferred as efficiently as possible, may be a good approximation to the actual state of affairs. The ratio of Kh/Km decreases as the thermal stratification becomes more unstable, which effect is opposite to what should occur. What this means, however, is that the upper bound on the stress is too large at large Rayleigh numbers.

Another result of geophysical interest is the transition Richardson number between free and forced convection. In Figures 4 and 5, upper bounds on the heat flux and stress have been plotted as a function of $Ra\ R^{-2}$. These calculations were carried out for Rayleigh numbers beyond 10^9 in order to attain values of $Ra\ R^{-2}$ of order 10^{-1} for Reynolds numbers of order 10^6.

If a Richardson number is defined as $-Ra\ R^{-2}\ Pr^{-1}$, then apart from a factor of 1.4, Figures 4 and 5 represent a plot of heat flux and stress as a function of Richardson number. Priestly's (1959) transition Richardson number of -.03 corresponds to approximately .02 on the scales presented here. Figures 4 and 5 indicate a gradual transition between two different regimes. Additional field data, grouped according to Reynolds number, might also indicate a more gradual transition than Priestly's (1959) curves indicate.

5. Conclusion

Upper bounds on the heat flux and stress have been obtained for the combined problem of Rayleigh convection and plane Couette flow. The bounding procedures described in this paper ought to be of use in other contexts such as a system involving heat, momentum, and mass transfer.

The prediction of the ratio of eddy exchange coefficients in near neutral conditions, and the identification of two distinct Richardson number regimes indicates that the extremal approach (originally put forth by Howard) will play an important role in our understanding of geophysical phenomena.

Improvements in the estimation techniques and the specification of additional integral constraints should result in upper bounds that are more in agreement with experimental data.

Acknowledgments

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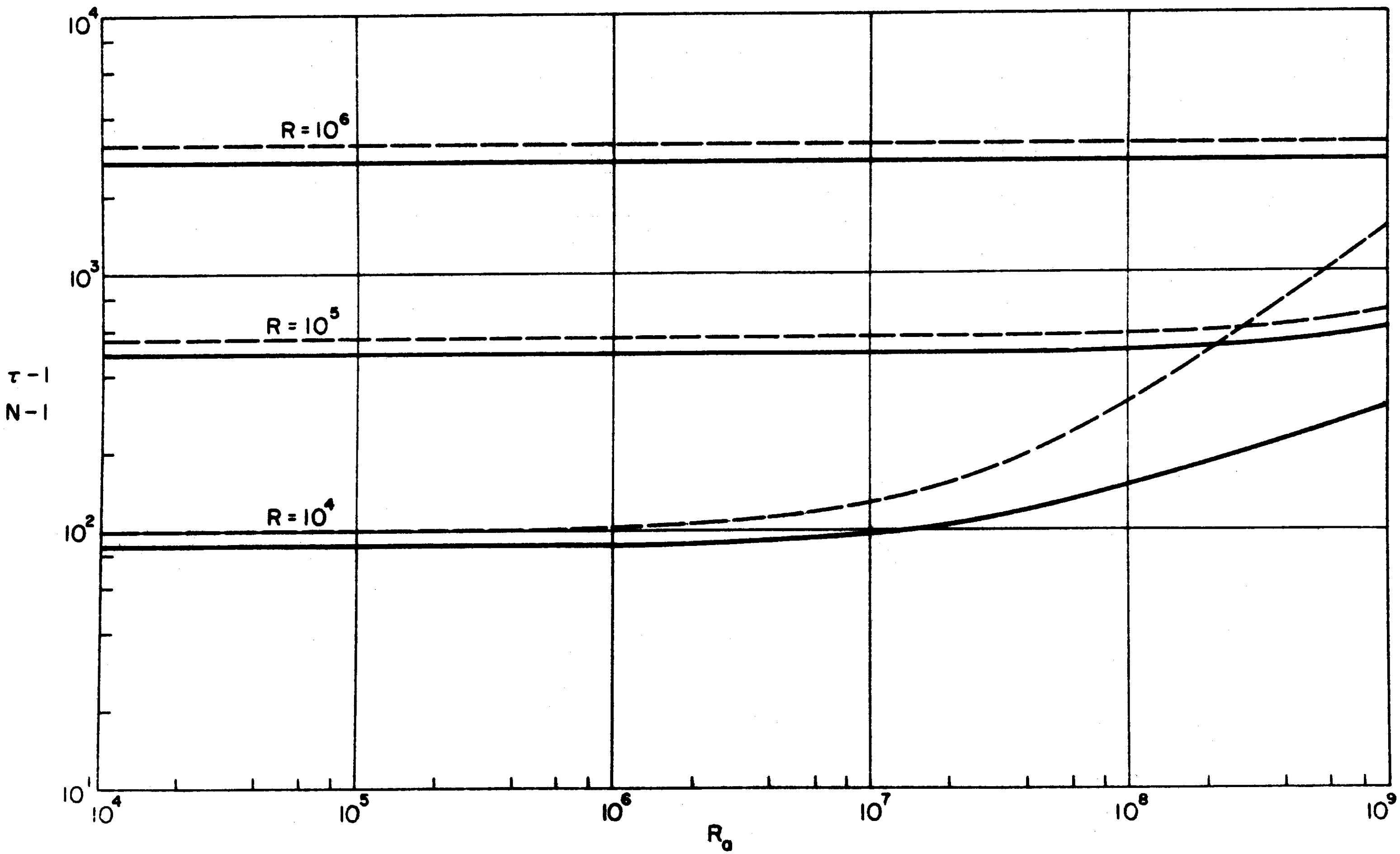

Fig. 1. Upper bounds on the Nusselt number and stress in terms of Rayleigh and Reynolds numbers (Pr=.7). The stress is given by the dashed curves and the Nusselt number by the solid curves.

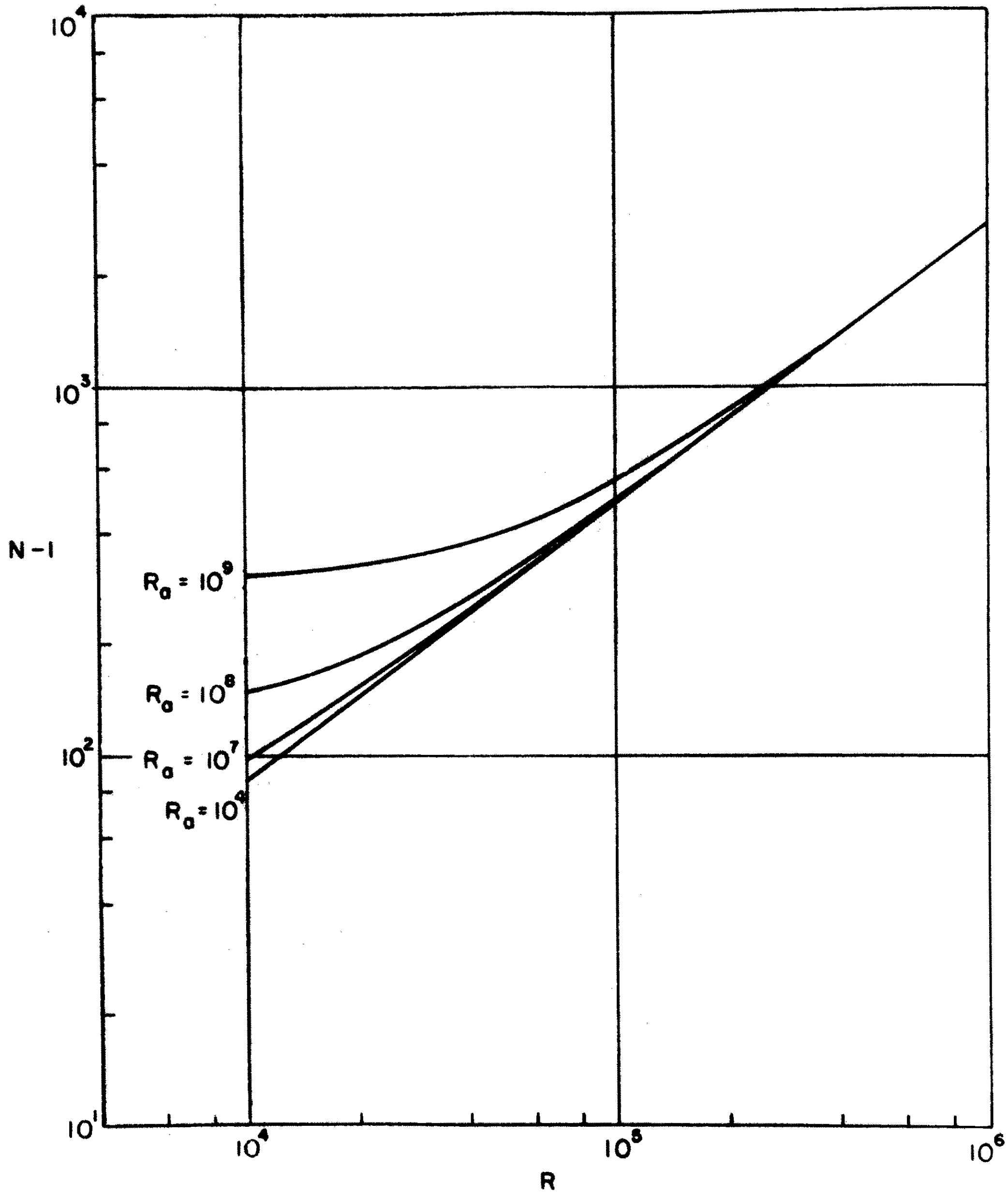

Fig. 2. Upper bounds on the Nusselt number in terms of Rayleigh and Reynolds numbers (Pr=.7)

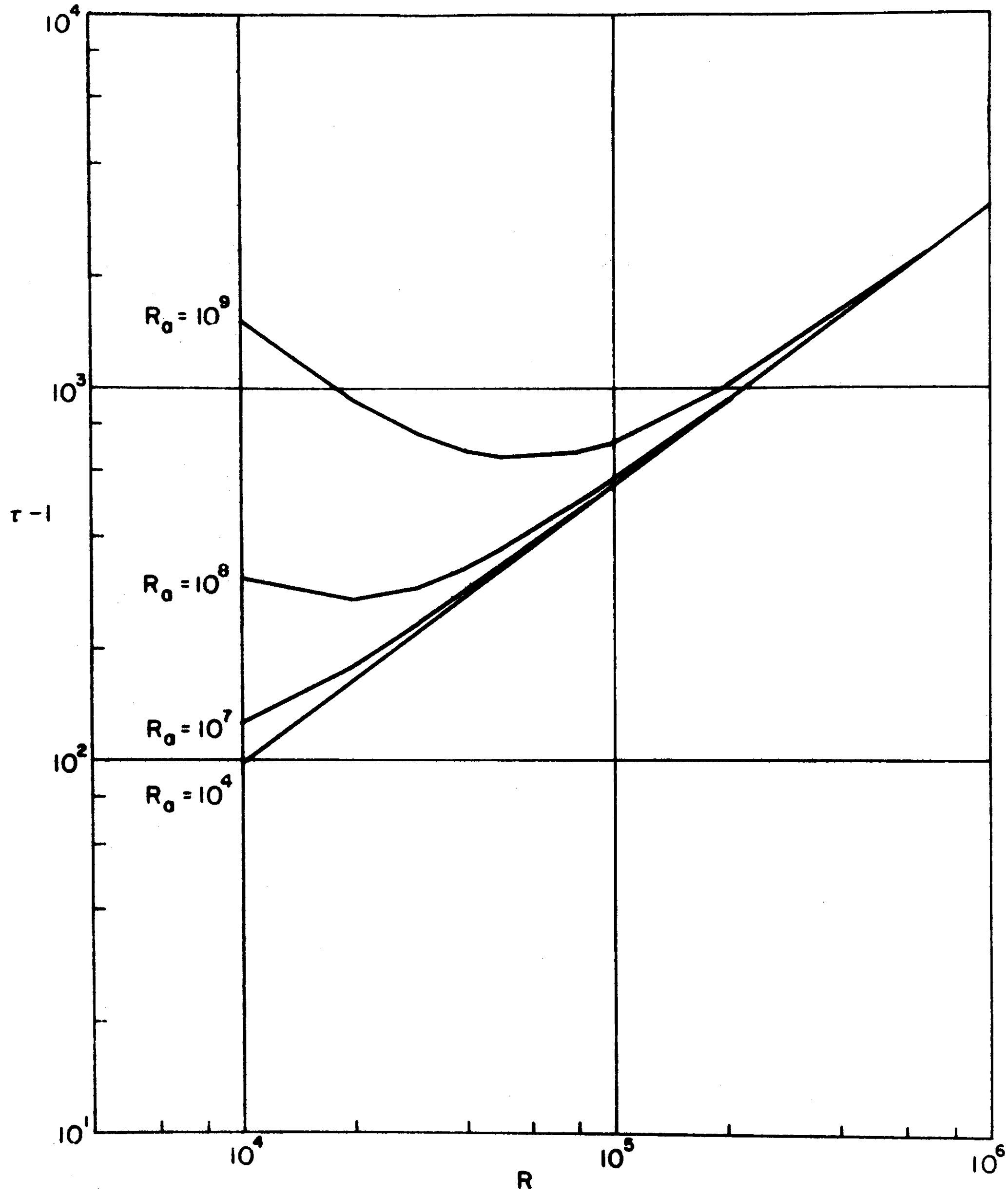

Fig. 3. Upper bounds on the stress in terms of Rayleigh and Reynolds numbers (Pr=.7)

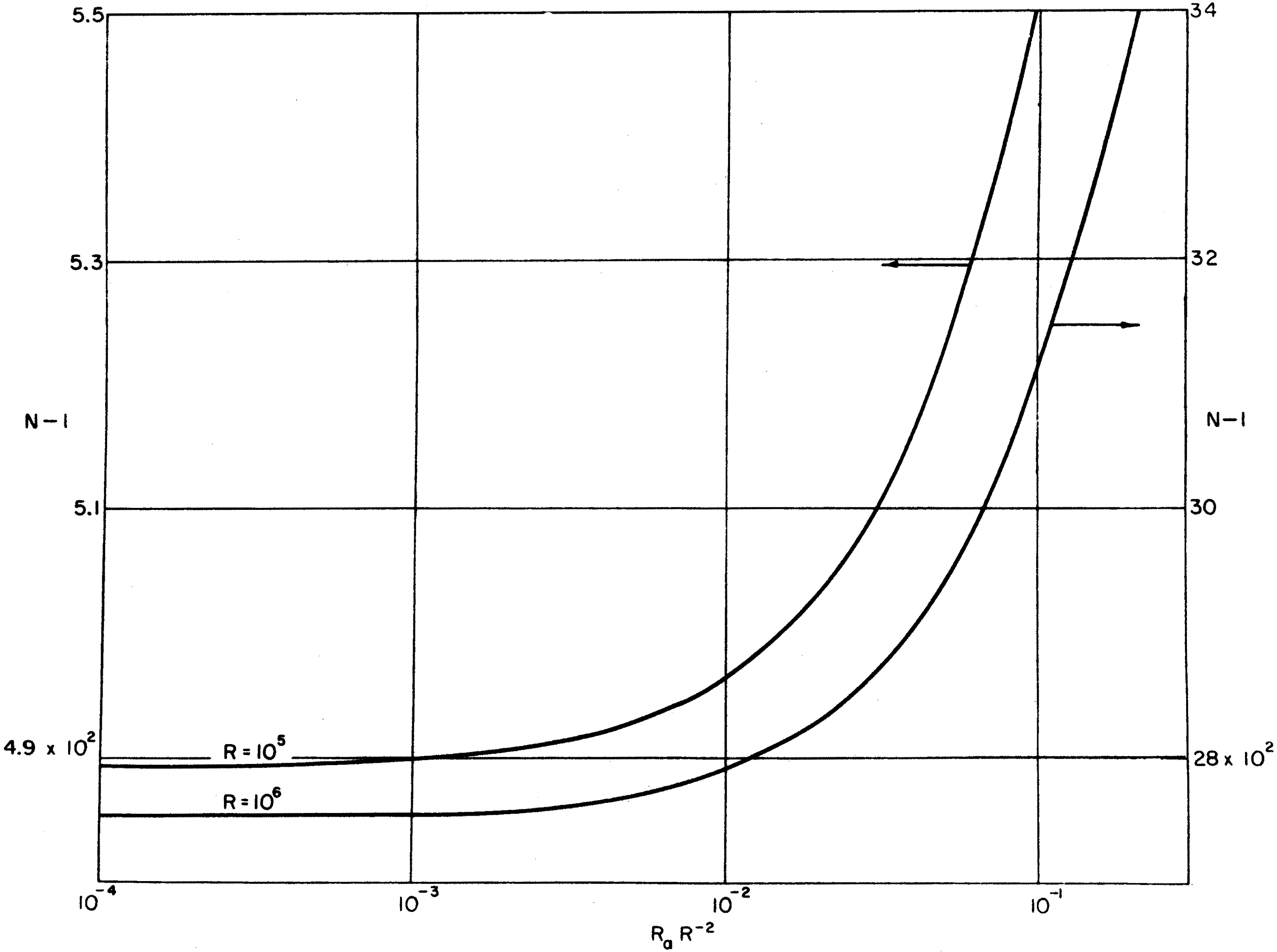

Fig. 4. Upper bounds on the Nusselt number in terms of Richardson and Reynolds numbers (Pr=.7)

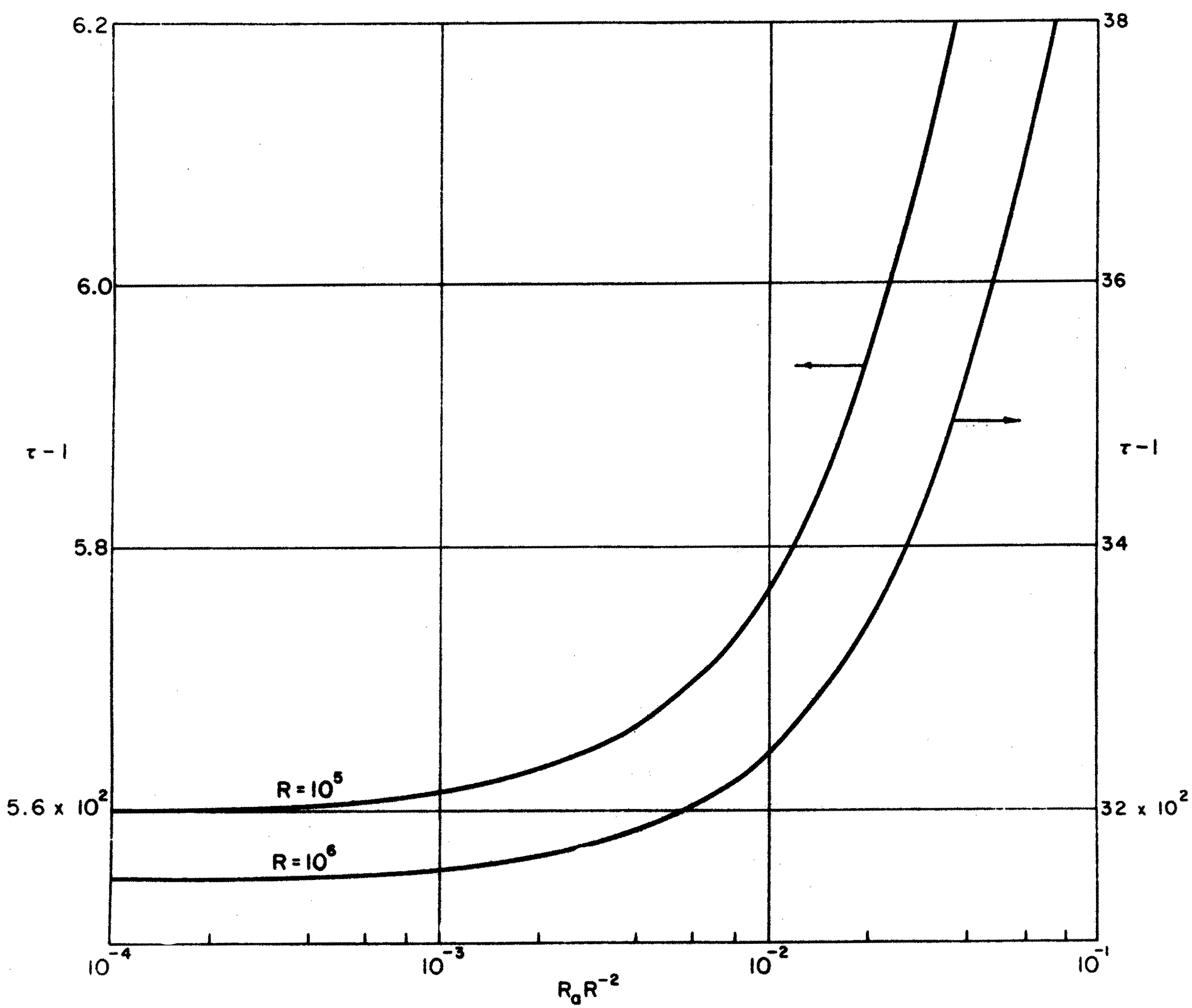

Fig. 5. Upper bounds on the stress in terms of Richardson and Reynolds numbers (Pr=.7)