DISSERTATION

DETERMINING SYNCHRONIZATION OF CERTAIN CLASSES OF PRIMITIVE GROUPS OF AFFINE TYPE

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ABSTRACT

DETERMINING SYNCHRONIZATION OF CERTAIN CLASSES OF PRIMITIVE GROUPS OF AFFINE TYPE

The class of permutation groups includes 2-homogeneous groups, synchronizing groups, and primitive groups. Moreover, 2-homogeneous implies synchronizing, and synchronizing in turn implies primitivity. A complete classification of synchronizing groups remains an open problem. Our search takes place amongst the primitive groups, looking for examples of synchronizing and nonsynchronizing. Using a case distinction from Aschbacher classes, our main results are constructive proofs showing that three classes of primitive affine groups are nonsynchronizing.

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Chapter 1

Introduction

One of the current topics of permutation group theory is that of synchronizing groups. The concept of synchronization started in automata theory and was later studied from a group-theoretic perspective [1]. In the study of transitive permutation groups, synchronizing groups are a class that lies between 2-homogeneous groups and primitive groups. Briefly, we say a group is synchronizing if there exists a partition with a transversal that is preserved by the action of the group (where a transversal is a set that intersects each cell of the partition exactly once), see Definition 3.4. Though primitive groups can be classified as a result of work by Michael O'Nan and Leonard Scott in 1979 that is now called the O'Nan–Scott Theorem, the same cannot be said for synchronizing groups. Since all synchronizing groups are primitive, the search begins in determining whether primitive groups are synchronizing. Here, determining classes of primitive groups which are always nonsynchronizing is just as valuable as finding classes which are always synchronizing.

Our early attempts at this problem involved searching through GAP's [2] extensive primitive group library with brute force to find synchronizing and nonsynchronizing groups of small degree. Amongst these results, we noticed an interesting pattern for the affine groups. Namely, several of the affine groups are nonsynchronizing in a similar way – as exhibited by an example of a specific partition and transversal. Furthermore, the partitions and transversals correspond to subspaces in the underlying vectorspace the group acts on. We therefore turn our attention to the linear part of the action, with matrix subgroups of $GL_n(p)$. To determine the structure of the possible subgroups, we refer to the Aschbacher Classes to create a case distinction. Our main results are constructing proper partitions and transversals to prove that groups from the $\mathcal{C}_2$, $\mathcal{C}_4$, and $\mathcal{C}_7$ classes are always nonsynchronizing, seen in Section 4.2, 4.4. and 4.7 respectively.

From the start we assume basic knowledge of groups, and begin at the analysis of groups as permutation groups by their actions on sets. All groups that we refer to will be finite, and act on finite sets. Thus, some of the statements we make do not hold for infinite groups, or groups acting on infinite sets. Although we speak strictly in terms of permutation groups, it is important to recall Cayley's Theorem: that every finite group is isomorphic to a subgroup of the symmetric group of some degree. Therefore any group can be thought of as a permutation group, and this is the viewpoint we will take, indicating where necessary the particular permutation action we choose.

Chapter 2

Preliminaries

2.1 Group Actions

Often times we care not only about groups as individual entities, but look at how they act on other mathematical objects. For this we will need the notion of *group actions*. Recall that S_Ω is the group of all permutations on the set Ω and S_n is the group of permutations on $\{1, 2, \ldots, n\}$.

Definition 2.1. Let Ω be a finite set and $G \leq S_\Omega$ a group. A *group action* of G on Ω is a map from $\Omega \times G \to \Omega$ given by $(\omega, g) \mapsto \omega^g$ satisfying $\omega^1 = \omega$ and $(\omega^g)^h = \omega^{gh}$ for all $\omega \in \Omega$ and all $g, h \in G$.

Permutation groups are often organized together based on the number of points on which they act.

Definition 2.2. Let $G \leq S_\Omega$. Then the *degree* of the action of G on Ω is the size of Ω, i.e. $|\Omega|$.

Considering the action of G on Ω as a whole, we also get a partition of Ω formed by the *orbits* of G on the elements of Ω.

Definition 2.3. Let G act on Ω with $\omega \in \Omega$. Then the *orbit of* ω is $\omega^G := \{\omega^g \mid g \in G\}$.

This brings us to a general consideration of how we can use some element in G to get from one point in Ω to another. The groups we are concerned with later will always have a way to get from one point to another, which gives us the concept of *transitivity*.

Definition 2.4. We say that a group action of G on Ω is *transitive* if for every $\alpha, \gamma \in \Omega$, there exists an element $g \in G$ such that $\alpha^g = \gamma$.

Alternatively, we notice that a group action is transitive if and only if it has a single orbit. It also turns out that we also have a similar, but more restrictive property that will be important as we proceed.

Definition 2.5. We say the action of G on Ω is *regular* if for any α, β in Ω, there exists exactly one element $g \in G$ such that $\alpha^g = \beta$.

We can thus say that a group action is regular if it is isomorphic to the left-action of the group on itself.

When G acts on Ω we have an inherited action of G on sets and tuples with elements from Ω. We define these actions pointwise, that is $(\omega_1, \omega_2, \dots, \omega_k)^g = (\omega_1^g, \omega_2^g, \dots, \omega_k^g)$ for tuples and $\{\omega_1, \omega_2, \dots, \omega_k\}^g = \{\omega_1^g, \omega_2^g, \dots, \omega_k^g\}$ for subsets. Furthermore we can talk about the orbits of sets, $T^G := \{T^g \mid g \in G\}$.

Example 2.6. The cycle $g = (1, 2, 3, 4) \in S_4$ acts on the set $\{1, 3\}$ by $\{1, 3\}^g = \{2, 4\}$ and the tuple $(1, 4)$ by $(1, 4)^g = (2, 1)$.

We can therefore extend the concept of transitivity as follows.

Definition 2.7. An action of G on Ω is *k-transitive* if for any pair of k-tuples of distinct elements, $(\alpha_1, \alpha_2, \dots, \alpha_k), (\gamma_1, \gamma_2, \dots, \gamma_k) \in \Omega^k$, there exists some $g \in G$ such that $(\alpha_1, \alpha_2, \dots, \alpha_k)^g = (\gamma_1, \gamma_2, \dots, \gamma_k)$.

Note that the requirement of distinct elements here lies within the tuples: i.e. $\alpha_i \neq \alpha_j$ and $\gamma_i \neq \gamma_j$ for all $i \neq j$.

Definition 2.8. Similarly, an action of G on Ω is *k-homogeneous* if for any two k-subsets of Ω, $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ and $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$, there exists $g \in G$ such that $\{\alpha_1, \dots, \alpha_k\}^g = \{\gamma_1, \dots, \gamma_k\}$.

Let $\Omega^{\{k\}}$ denote the set of k-element subsets of Ω. Then in other words, we can say that $G \leq S_\Omega$ is k-homogeneous if G acts transitively on $\Omega^{\{k\}}$. We then notice the hierarchical relationship between k-transitive and k-homogeneous.

Lemma 2.9. *If an action of G on Ω is k-transitive then it is also k-homogeneous.*

Proof. Let G be k-transitive on Ω. Take any two k-subsets on Ω, denoted $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ and $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$. By k-transitive, there exists $g \in G$ such that $(\alpha_1, \alpha_2, \dots, \alpha_k)^g = (\gamma_1, \gamma_2, \dots, \gamma_k)$. So $\alpha_i^g = \gamma_i$ for all $i = 1, \dots, k$. Then $\{\alpha_1, \dots, \alpha_k\}^g = \{\gamma_1, \dots, \gamma_k\}$. $\square$

In 1972, Kantor described all k-homogeneous groups G of degree n which are not k-transitive [3]. Namely, they must fall into one of the following categories, with $n \geq 2k$:

1. $k = 2$ and $G \leq A\Gamma L(1, n)$ with $n \equiv 3$ (mod 4)

2. $k = 3$ and $PSL(2, n-1) \leq G \leq P\Gamma L(2, n-1)$ with $n \equiv 0$ (mod 4)

3. $k = 3$ and $G = AGL(1, 8)$, $G = A\Gamma L(1, 8)$, or $G = A\Gamma L(1, 32)$

4. $k = 4$ and $G = PSL(2, 8)$, $G = P\Gamma L(2, 8)$, or $G = P\Gamma L(2, 32)$

Kantor's proof relies on the Feit–Thompson Theorem, as well as work published seven years previously by Livingstone and Wagner [4]. One interesting fact to note that originated from their paper is that each of these cases arises for $k \leq 4$. That is to say that for $k \geq 5$, k-homogeneous and k-transitive are equivalent properties. However, due to the classification of finite simple groups, only Mathieu groups, alternating groups, and symmetric groups are 5-transitive, and only the symmetric and alternating groups are k-transitive for $k > 5$.

Lemma 2.10. *If G is k-transitive on Ω for $k > 1$, G is also $(k-1)$-transitive on Ω.*

Proof. Suppose G is k-transitive. Take any two tuples $(\alpha_1, \ldots, \alpha_{k-1}), (\gamma_1, \ldots, \gamma_{k-1}) \in \Omega^{k-1}$. Since G is k-transitive, there exists $g \in G$ such that $(\alpha_1, \ldots, \alpha_{k-1}, \delta)^g = (\gamma_1, \ldots, \gamma_{k-1}, \delta^g)$, and thus $(\alpha_1, \ldots, \alpha_{k-1})^g = (\gamma_1, \ldots, \gamma_{k-1})$. $\square$

To make an analogous statement about k-homogeneity, we require a few extra lemmata. The first result is a complementary property, with a k-homogeneous group also being $(|\Omega| - k)$-homogeneous.

Lemma 2.11. *If G is k-homogeneous on Ω, G is also r-homogeneous on Ω for $r = |\Omega| - k \neq 0$.*

Proof. Suppose G is k-homogeneous. Let $S, T \subseteq \Omega$ with $|S| = |T| = r$. Since G is k-homogeneous, there exists some $g \in G$ such that $(\Omega - S)^g = (\Omega - T)^g$. Then $S^g = T^g$ and therefore G is r-homogeneous. $\square$

The next two lemmata from Dixon and Mortimer [5] concern counting orbits to give us the rest of the pieces we need.

Lemma 2.12 ([5] Lemma 9.4A)**.** *Let G be a group which acts on both Γ and Δ but leaves some subset $\Lambda \subseteq \Gamma \times \Delta$ invariant. Let $V = \{f\colon \Gamma \to \mathbb{Q} \mid f(\gamma) = f(\gamma^x)\, \text{for all } x \in G\}$ and $W = \{f\colon \Delta \to \mathbb{Q} \mid f(\delta) = f(\delta^x)\ \text{for all } x \in G\}$. If the $\mathbb{Q}$-linear transformation $\theta : V \to W$ defined by $f^\theta(\delta) := \sum_{(\gamma,\delta)\in\Lambda} f(\gamma)$ is injective, then the number of orbits of G on Δ is at least the number of orbits of G on Γ.*

Proof. By definition of V and W, their dimensions are the number of orbits of G on the sets Γ and Δ respectively. Since Λ is G-invariant, then θ maps V into W. Then if θ is injective, we have $\dim(W) \geq \dim(V)$ and therefore the number of orbits of G on Δ is at least the number of orbits of G on Γ. □

Lemma 2.13 ([5] Theorem 9.4A)**.** *Let $G \leq S_\Omega$ with integers $0 \leq m \leq k$ such that $m + k \leq |\Omega|$. Then G has at least as many orbits on $\Omega^{\{k\}}$ as it has on $\Omega^{\{m\}}$.*

Proof. If we let $\Gamma = \Omega^{\{m\}}$, $\Delta = \Omega^{\{k\}}$ and $\Lambda = \{(T, S) \in \Omega^{\{m\}} \times \Omega^{\{k\}} \mid T \subseteq S\}$, we can apply Lemma 2.12. Clearly G acts on $\Omega^{\{m\}}$ and $\Omega^{\{k\}}$, and leaves Λ invariant since $S \subseteq T \implies S^x \subseteq T^x$ for subsets $S, T \subseteq \Omega$, for all $x \in G$. Then we need to show that θ is injective, which we will do by showing it has a trivial kernel. Suppose that for some $f : \Omega^{\{m\}} \to \mathbb{Q}$,

$$0 = f^\theta(S) = \sum_{T\in\Omega^{\{m\}},\ T\subseteq S} f(T) \quad \text{for all } S \in \Omega^{\{k\}}.$$

For any subsets $R \subseteq P \subseteq \Omega$ let

$$g(R, S) = \sum_{T\in\Omega^{\{m\}},\ R\subseteq T\subseteq P} f(T).$$

Then we note that $g(\varnothing, P) = 0$ when $|P| = k$ by assumption. And if $|P| > k$, we can apply the fact that

$$\binom{|P|}{k} \cdot g(\varnothing, P) = \sum_{T \subseteq P,\ |T|=k} g(\varnothing, T)$$

where each summand is 0, and therefore we can conclude $g(\varnothing, P) = 0$ whenever $|P| \geq K$. Next we note that for any $\omega \in \Omega$, $g(R, P) = g(R - \{\omega\}, P) - g(R - \{\omega\}, P - \{\omega\})$. Then we can induct on $|R|$ to conclude that $g(R, P) = 0$ for any $R \subseteq P$ when $|P - R| \geq k$. Then for $T \in \Omega^{\{m\}}, S \in \Omega^{\{k\}}$ with $T \subseteq S$, we get $f(T) = g(T, S) = 0$. Therefore, $\ker(\theta) = 0$, so by Lemma 2.12, G has at least as many orbits on $\Omega^{\{k\}}$ as it has on $\Omega^{\{m\}}$. $\square$

Now we have the tools to show that k-homogeneity is inherited for decreasing k.

Lemma 2.14. *If G is k-homogeneous on Ω for $k > 1$, G is also $(k-1)$-homogeneous on Ω.*

Proof. Without loss of generality by Lemma 2.11, let G be k-homogeneous on Ω for $0 < 2k \leq |\Omega| + 1$. Then $(k-1) + k \leq |\Omega|$. So by Lemma 2.13, G has as at least as many orbits on $\Omega^{\{k\}}$ as it has on $\Omega^{\{k-1\}}$. But k-homogeneity implies that G is transitive on k-sets, and therefore has exactly one orbit on $\Omega^{\{k\}}$, and thus has no more than one orbit on $\Omega^{\{k-1\}}$, but cannot have 0 orbits. Therefore G is $(k-1)$-homogeneous. $\square$

Then since k-transitivity implies k-homogeneity, and both properties are inherited down for smaller values of k, they can all be reduced to 2-homogeneity.

Corollary 2.15. *If G is k-homogeneous or k-transitive for any $k > 1$, Lemmata 2.9, 2.10, and 2.14, imply that G is 2-homogeneous.*

Thus, 2-homogeneous is the property that we will mainly be concerned about, and one of the key pieces in our journey to define synchronizing groups. The other key piece primitivity.

2.2 Block Systems and Primitivity

The actions of groups on sets of points can be further studied by considering the action on *sets of sets* of points: partitions. In particular, we can see if partitions are preserved by the group.

Definition 2.16. Let G act transitively on Ω. A *block system* for this action is a partition $\mathcal{B}$ of Ω that is invariant under the action of G. That is to say for each $B \in \mathcal{B}$ and any $g \in G$, $B^g = B'$ for some $B' \in \mathcal{B}$. We call any $B \in \mathcal{B}$ a *block*.

Note that any transitive action has two *trivial* block systems, namely $\{\Omega\}$ and $\big\{\{\omega\} \mid \omega \in \Omega\big\}$. After seeing some examples, one might notice that the preserved partitions display a symmetry, in that the cells must be of the same size.

Definition 2.17. We say that a partition $\mathcal{P}$ of a set is *uniform* if each cell is of equal size.

Furthermore, it turns out our observation following the definition of block systems does indeed hold.

Lemma 2.18. *A block system must be uniform.*

Proof. Let $\mathcal{B}$ be a block system for $G \leq S_\Omega$ with arbitrary distinct cells $A, B \in \mathcal{B}$. Then for $a \in A$ and $b \in B$, there exists some $g \in G$ such that $a^g = b$ by transitivity. Then $A^g = B$ since A, B are part of a block system. Then $|A| = |B|$ and therefore the partition is uniform. $\square$

We will revisit uniformity later for synchronizing groups, but need to understand *primitivity* first.

Definition 2.19. We say an action of G on Ω is *primitive* if the only block systems it affords are the trivial ones. Otherwise, we say the action is *imprimitive*.

Moreover, if the action of $G \leq S_\Omega$ is primitive, we say that G is a primitive group. To understand this property, we will give a few examples.

Example 2.20. For $n > 2$, S_n and A_n act primitively on the set with n elements.

S_n and A_n have this property because they move points maximally (being the largest groups of degree n).

Example 2.21. Consider the action of the dihedral group D_{12} on six vertices (labelled consecutively). Then

$$\{\{1,3,5\},\{2,4,6\}\}$$

is a block system for this action, and therefore the standard action of D_{12} is imprimitive. Namely, a rotation exchanges $\{1,3,5\}$ with $\{2,4,6\}$ while a reflection holds them the same.

A great tool for examining block systems (and therefore determining whether a group is primitive) comes from the fact that we do not need to know the entire partition.

Lemma 2.22. *A block system is determined uniquely by one of its blocks.*

Proof. Let $\mathcal{B}$ be a block system for G acting transitively on Ω with some block B. Let $|\Omega| = n, |B| = k$. Consider B^g for some $g \in G$. By definition, B^g is also a block in $\mathcal{B}$. Then for any $\alpha \in B$, if $\alpha^g \in B$ then $B^g = B$. Else if $\alpha \notin B$, we must have that B, B^g are disjoint. Since G acts transitively on Ω we have that $\{B^g \mid g \in G\}$ will cover Ω, and therefore give the entire block system. $\square$

We will use the previous lemma without name often as an inherent property of block systems. A further helpful fact about primitivity is that it is inherited upwards for parent groups – as the preservation of block systems is inherited downward to subgroups.

Lemma 2.23. *If G acts primitively on Ω and $G \leq H \leq S_\Omega$ then H also acts primitively on Ω.*

Proof. Suppose for a contradiction that H did not act primitively on Ω. Then there exists some nontrivial block system $\mathcal{B}$ that is preserved by H. Since $G \leq H$, then G must also preserve $\mathcal{B}$. Then G could not be primitive, a contradiction. $\square$

Finally is an interesting and profound result, relating our previous properties of k-transitivity and k-homogeneity with that of primitivity. This is a key factor in the study of synchronizing groups.

Lemma 2.24. *If the action of G on Ω is k-homogeneous or k-transitive for $k > 1$, G acts primitively on Ω.*

Proof. Note that if G is k-homogeneous or k-transitive for $k > 1$, then by Corollary 2.15, G is 2-homogeneous. In search of a contradiction, suppose that G had B, a nontrivial block in a block system and therefore G were not primitive. Let $\alpha, \beta \in B$ and $\gamma \notin B$. Since G is 2-homogeneous, there exists some $g \in G$ such that $\{\alpha, \beta\}^g = \{\alpha, \gamma\}$. But then $B = B^g$ since α can only be in one block, so $\gamma \in B$, which gives a contradiction. □

We thus have that the class of 2-homogeneous groups is a subclass of primitive groups. Our ultimate goal is to describe a class that lies between these two: synchronizing groups.

2.3 Wreath Products

An additional group structure that is important to understand as we proceed is that of wreath products. Specifically for the case of primitivity, we care about wreath products with the *product action*, which differs from an imprimitive wreath action.

Definition 2.25. Let $H \leq S_\Gamma$ and $K \leq S_\Delta$ be groups acting on Γ and Δ respectively. Then we say $G = H \wr K$ is the *wreath product of* H *and* K which acts on Γ^Δ.

Here, the group has the form of $(H \times H \times \cdots \times H) \rtimes K$ with $|\Gamma|$ copies of H, where K acts by permuting the copies of H.

Definition 2.26. Let $H \leq S_\Gamma$ and $K \leq S_\Delta$ be groups acting on Γ and Δ respectively. Then we say $G = H \wr K$ acts on Γ^Δ with the *product action*.

It turns out that these groups act primitively under a specific set of circumstances.

Lemma 2.27 (Dixon & Mortimer [5] Lemma 2.7A)**.** *The product action of* $H \wr K$ *on* Γ^Δ *is primitive if and only if* H *is primitive and not regular on* Γ*,* Δ *is finite, and* K *is transitive on* Δ*.*

2.4 Simple Groups and the O'Nan–Scott Theorem

Lastly, we will refer to the O'Nan–Scott Theorem as given in [5]. As the theorem essentially gives us a list of boxes that primitive groups fall into, we will use these boxes in the following chapter in our search for nonsynchronizing primitive groups. We'll start by recalling the definition of a simple group.

Definition 2.28. We say that a nontrivial group G is *simple* if its only normal subgroups of G are the trivial group and G itself.

Extending the idea with normal subgroups, we have a special subgroup called the *socle*.

Definition 2.29. The *socle* of a group G, denoted $\mathrm{Soc}(G)$, is the subgroup generated by all minimal normal subgroups of G.

Since distinct minimal normal subgroups intersect trivially, we get a commutativity result between them.

Lemma 2.30. *Distinct minimal normal subgroups of a group G commute with each other.*

Proof. Let H, N be distinct minimal normal subgroups of G, thus $H \cap N = \langle () \rangle$. Let $h \in H, n \in N$. Then $nhn^{-1} \in H$ and $hn^{-1}h^{-1} \in N$. Then

$$(nhn^{-1})h^{-1} \in H \qquad \text{and} \qquad n(hn^{-1}h^{-1}) \in N.$$

Thus $nhn^{-1}h^{-1} \in H \cap K$, so $nhn^{-1}h^{-1} = ()$, and thus $nh = hn$. Therefore, distinct minimal normal subgroups commute with each other. $\square$

Together the commutativity from Lemma 2.30 and the generating structure of the socle by Definition 2.29, we inherit a direct product structure on the subgroup.

Corollary 2.31. *The subgroup $Soc(G)$ is a direct product of distinct minimal normal subgroups of G.*

The above corollary follows since a group generated by commuting subgroups has the structure as a direct products of the subgroups. We now have the tools to define *almost simple*.

Definition 2.32. We say that G is *almost simple* if $\mathrm{Soc}(G)$ is simple.

Alternatively, one might say that G is almost simple if G lies between a simple group, H and the automorphism group of H, i.e. $H \leq G \leq \text{Aut}(H)$. We will lastly note the necessary structure of minimal normal subgroups from [5].

Lemma 2.33. *Every minimal normal subgroup of G is a direct product of isomorphic simple groups.*

And finally we can use the prior definitions and lemmata to present the O'Nan–Scott Theorem, the tool we need to classify the primitive groups

Theorem 2.34. (The O'Nan–Scott Theorem) *Let G be a primitive group of degree n such that Soc$(G) = H$. Then one of the following is the case:*

1. *H is a regular elementary abelian p-group for p prime, $|H| = p^m = n$, and G is isomorphic to a subgroup of the affine group $AGL_m(p)$.*
2. *H is isomorphic to a direct product T^m of a nonabelian simple group T with one of the following:*
   - *(a) $m = 1$ and G is isomorphic to a subgroup of Aut(T).*
   - *(b) $m \geq 2$ and G is a subgroup of diagonal type with $n = |T|^{m-1}$.*
   - *(c) $m \geq 2$ and there exists some proper divisor d of m and some primitive group U such that Soc(U)$\cong T^d$, and G is isomorphic to a subgroup of $U \wr S_{m/d}$ with the product action, and $n = l^{m/d}$ where l is the degree of U.*
   - *(d) $m \geq 6$ with H regular and $n = |T|^m$.*

In a nutshell, the O'Nan–Scott Theorem tells us that every primitive group can be placed into (at least) one of five boxes - though some other sources choose their boxes slightly differently. This classification will be important for us as we look for synchronizing groups within these boxes. Our research will focus on part 1, the *affine type*, covered in Chapter 4, and we will review the progress that has been made on part 2 in Chapter 5.

Chapter 3

Synchronizing Groups

Within primitive groups lies the lesser studied class of permutation groups known as synchronizing groups. The concept of synchronizing originated from the study of Automata Theory [1]. There, the concept concerns the existence of a "reset word" in some finite state automata. As automata are maps between states, this concept transferred directly to monoids with a monoid being synchronizing when it contains an element h such that $|\mathrm{im}(h)| = 1$. A group G is then synchronizing if $\langle G, f\rangle$ is a synchronizing monoid for some map f that is not a permutation. In 2009 Peter Neumann refined this concept of a synchronizing group to a concept in permutation group theory regarding section-regular partitions [6]. Acting from a group theoretic mindset, we will treat this property (Definition 3.4) as our definition for a synchronizing group. Building toward a classification of synchronizing groups is the current goal.

3.1 Section-regular Partitions

The first step to understanding the group theoretic definition of synchronization will be understanding section-regular partitions, so we start with the definition of a section.

Definition 3.1. Let P be a partition of Ω. We call a subset $T \subseteq \Omega$ a *section* for P if T contains exactly one element from each cell of P. Alternatively, we may say that T is a *transversal* of P.

Let us consider an example before defining section-regular partitions.

Example 3.2. Consider the partition $P = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$. Then $\{2, 7, 9\}$ is a section of P, as is $\{1, 8, 12\}$. However, $\{1, 2, 5, 9\}$ is not, since it contains two elements from the first cell, and $\{1, 9\}$ is not because it does not contain any elements from the third cell.

The preservation of a section under a group action will give us the concept of section-regular.

Definition 3.3. Let $G \leq S_\Omega$ and P a partition of Ω. We say that P is a *section-regular partition* for G if there exists a section T of P such that T^g is a section of P for any $g \in G$.

That is to say, G is section-regular if the group action yields an orbit of sections for the partition. Now we have the tools to define synchronizing.

Definition 3.4. Let $G \leq S_\Omega$. We say that a group G is *synchronizing* if it affords no nontrivial section-regular partition of Ω. We say that G is *nonsynchronizing* otherwise.

If we observe the similarities between a section-regular partition under a group, and a group preserving a block system, we can see the relation between synchronization and primitivity.

Theorem 3.5. *A synchronizing group is primitive.*

Proof. We will prove the contrapositive; that $G \leq S_\Omega$ having a nontrivial block system implies G has a nontrivial section-regular partition. Let B be a nontrivial block system for G. Pick T to be section for B. Then since each $t \in T$ are in distinct cells, then for all $t \in T, g \in G$ each of the t^g must also be in distinct cells since G preserves the partition. Then B is a nontrivial section-regular partition for G. □

As we see the implications of comparing block systems with section-regular partitions when comparing primitive groups to synchronizing groups, we can consider further properties. We recall that Lemma 2.18 gave us that block systems must be uniform. Though the proof is not quite as straightforward, the analogous fact does hold that section-regular partitions must also be uniform.

Lemma 3.6. *A section-regular partition under a transitive group must be uniform [6].*

Proof. Let $\mathcal{P}$ be a section-regular partition of Ω under G, with section T. Let $\mathcal{P}$ have k cells denoted $P_1, \ldots, P_k$ with sizes $p_1, \ldots, p_k$ respectively. Without loss of generality, suppose $p_1 \leq p_2 \leq \cdots \leq p_k$. Then

$$|\Omega| = p_1 + p_2 + \cdots + p_k \geq kp_1.$$

Moreover, the equality $|\Omega| = kp_1$ holds exactly when $p_i = p_j$ for all i, j, which is to say when $\mathcal{P}$ is uniform.

Let the elements of T be $t_1, \ldots, t_k$ where $T \cap P_i = t_i$ for each cell P_i. Define H_i to be the stabilizer of t_i in G. Since $\mathcal{P}$ is section-regular with T, for any $g \in G$, $T^g \cap P_1$ has exactly a single element, namely t_1^g. Define

$$K_i := \bigcup_{x \in P_1} \{g \in G \mid t_i^g = x\} \text{ and thus } G = \bigcup_{i=1}^{k} K_i$$

So G is a union of k different sets, each K_i being a union of p_1 cosets of H_i. Since G is transitive, $|H_i| = |G|/|\Omega|$ for each i by the Orbit-Stabilizer Theorem. Therefore

$$|G| \leq k \cdot p_1 \cdot \frac{|G|}{|\Omega|} \leq k \cdot p_1.$$

Hence the equality $|G| = kp_1$ holds, implying that $\mathcal{P}$ is uniform.

□

Another property that synchronization shares with primitive groups is inheritance from subgroups.

Lemma 3.7. *If the action of G on Ω is synchronizing and $G \leq H \leq S_\Omega$ then the action of H is also synchronizing.*

Proof. Suppose for a contradiction that H is nonsynchronizing. Then there exists some section-regular partition P with section T such that T^h is a section for P for all $h \in H$. Then T^g must is a section for all $g \in G \leq H$. Then G could not be synchronizing, a contradiction. □

As the above is an analogous statement to Lemma 2.23, we can also show a stronger version of Lemma 2.24.

Theorem 3.8. *A 2-homogeneous group is synchronizing.*

Proof. Let G be 2-homogeneous and suppose G is not synchronizing for a contradiction. Then there exists a nontrivial section-regular partition P for G with section T. Let B and C be two cells of P intersecting T with t_B and t_C respectively. Let $b \neq t_B$ be another element of B. Then since G

is 2-homogeneous, there exists a $g \in G$ such that $\{t_B, t_C\}^g = \{t_B, b\}$. But since $\{t_B, t_C\}^g$ should be a subset of a section, we have a contradiction since 2 elements of B cannot be in a section. Therefore G must be synchronizing. $\square$

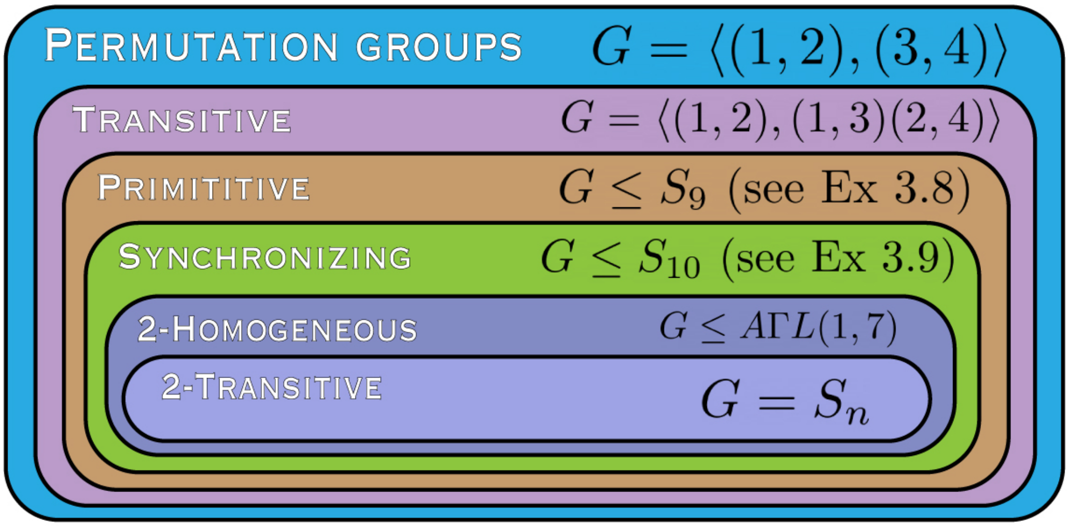

Figure 3.1: Hierarchy of permutation groups from transitive to 2-transitive with examples of each level.

This gives us that synchronizing is a class of permutation groups that lies between 2-homogeneous and primitive as illustrated in Figure 3.1. Therefore a classification of synchronizing groups will require an examination of primitive groups that are not synchronizing, and synchronizing groups that are not 2-homogeneous. We will look at a few of the smallest such examples.

3.2 Examples

Example 3.9. The group $G := \langle (2,7,3,4)(5,8,9,6), (1,2,3)(4,5,6)(7,8,9) \rangle \leq S_9$ is primitive but not synchronizing.

Proof. First note that G is primitive group with abelian socle. Namely,

$$\mathrm{Soc}(G) = \langle (1,3,2)(4,6,5)(7,9,8), (1,7,4)(2,8,5)(3,9,6) \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$$

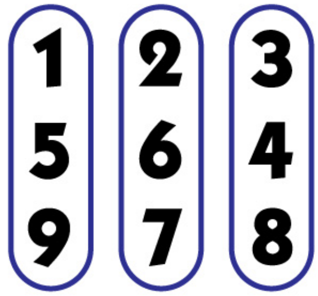

Figure 3.2: Partition for Ex. 3.9.

To show that G is not synchronizing, consider the partition $\{\{1,6,9\},\{2,6,7\},\{3,4,8\}\}$ (as shown in Figure 3.2) with section $\{1,2,3\}$. Note the orbit

$$\{1,2,3\}^G = \{\{1,2,3\},\{1,4,7\},\{2,5,8\},\{7,8,9\},\{3,6,9\},\{4,5,6\}\}$$

with each set in the orbit also a section for the given partition. Therefore G is not synchronizing.

$\square$

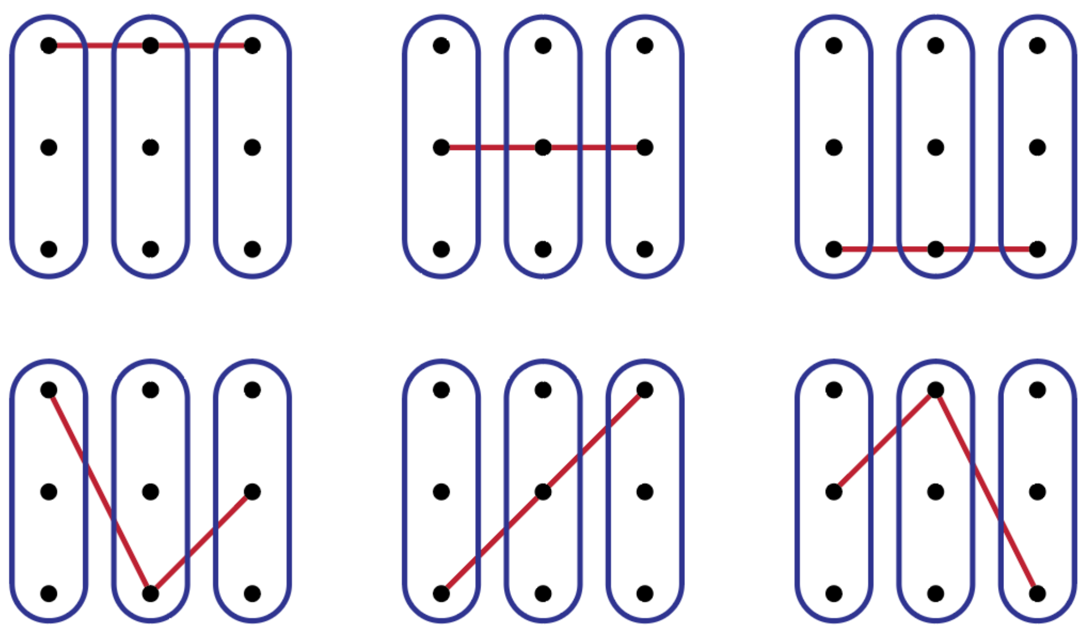

Figure 3.3: Example of partition with all images of the given section from Example 3.9 with the points labeled as in Fig. 3.2

We've thus shown that synchronization is strictly stronger than primitive, and next will show that 2-homogenous is strictly stronger than synchronization.

Example 3.10. Consider the action of S_5 on the 2 element subsets of $\{1, 2, 3, 4, 5\}$. This is a synchronizing action that is not 2-homogeneous.

Proof. To show this group is not 2-homogeneous, note there exists no permutation that maps $\{\{1, 2\}, \{1, 3\}\} \mapsto \{\{1, 2\}, \{4, 5\}\}$ because if $\{1, 2\}^g = \{1, 2\}$ then $\{1, 3\}^g$ couldn't equal $\{4, 5\}$. Alternatively if $\{1, 2\}^g = \{4, 5\}$ then $\{1, 3\}^g$ wouldn't be $\{1, 2\}$.

To show this group is synchronizing, we solicit some help from GAP [2]. We will consider this group as $H \leq S_{10}$ acting on the ten subsets which we from here relabel as $\{1, \ldots, 10\}$. In search of a nontrivial section-regular partition, we will apply Lemma 3.6 to consider the cases of having two cells of size 5 or five cells of size 2.

Case 1: First we consider partitions with 2 cells of size 5. Therefore a section for this partition would be a subset of size 2, of which there are 45. We look at the orbits of these subsets and see that there is an orbit of size 30 containing the section $\{1, 2\}$ and an orbit of size 15 containing the section $\{1, 8\}$. First let's consider the action of the $H_1 := Stab_H(1)$ on {1,2}. Note that $2^{H_1} = \{2, 3, 4, 5, 6, 7\}$. But we can't have that $\{1, q\}$ is a section for any $q \in \{2, 3, 4, 5, 6, 7\}$ as that would imply we had a cell of size at least 6. Then we must have $\{1, q\}$ a section for $q \in \{8, 9, 10\}$. Then we consider the orbits of 1 under the stabilizers of $8, 9, 10$ and see that $1^{H_8} = \{1, 4, 7\}$, $1^{H_9} = \{1, 3, 6\}$, and $1^{H_{10}} = \{1, 2, 5\}$. Then for $\{1, q\}$ to be a section for $q \in \{8, 9, 10\}$ we would have to have 1 to be in a cell of size at least 7 (including $\{1, 2, 3, 4, 5, 6\}$), which couldn't be a cell for a uniform partition, giving us a contradiction.

Case 2: Now we consider partitions with 5 cells of size 2 which therefore have sections of size 5. There are 252 possible sections to consider, which fall into 6 orbits under H. We will consider $\{1, 2, 3, 4, 5\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 5, 7\}, \{1, 2, 3, 5, 10\}, \{1, 2, 3, 7, 9\}, \{1, 2, 6, 9, 10\}$ as the representatives for each of these orbits.

If $\{1, 2, 3, 4, 5\}$ were a section for a partition, that means that each of the 5 cells of the partition must contain one of $\{1, 2, 3, 4, 5\}$. Then we consider the orbits of each of these under H_1 and note that $2^{H_1} = 3^{H_1} = 4^{H_1} = 5^{H_1} = \{2, 3, 4, 5, 6, 7\}$. Therefore if $\{1, 2, 3, 4, 5\}$ is a section for a partition, then: 1, 2, 3, 4, and 5 must all be in different cells, 2 cannot share a cell with 6 or 7, 3 cannot share a cell with 6 or 7, 4 cannot share a cell with 6 or 7, and 5 cannot share a cell with 6 or 7. Therefore the only candidate for a cell containing 6 or 7 is that which also contains 1. Then we would have a cell of size at least 3, implying the partition isn't uniform, giving us a contradiction.

If $\{1, 2, 3, 5, 6\}$ were a section for a partition, we have the analogous argument that neither 4 nor 7 could share a cell with any of 2, 3, 5, or 6. Then they would both have to share a cell with 1, giving us a contradiction to uniformity.

The same argument applies if $\{1, 2, 3, 5, 7\}$ were a section for a partition, concluding that 4 and 6 would have to share a cell with 1, for a contradiction to uniformity.

If we suppose $\{1, 2, 3, 5, 10\}$ were a section for a partition, we must analyze an additional point stabilizer for a different approach. Here we recall that neither 6 nor 7 can share a cell with 2, 3, or 5 since they are in the same orbit under H_1. Now we apply the fact that the orbit $10^{H_2} = \{6, 7, 10\}$ to reach the conclusion that additionally, 6 nor 7 can share a cell with 10, implying they share a cell with 1 for a contradiction.

Similarly for $\{1, 2, 3, 7, 9\}$ we note that 4 or 5 cannot share a cell with 2, 3 or 7 since they are in the same orbit under H_1. We then use the orbit $9^{H_2} = \{1, 3, 4, 5, 8, 9\}$ to say that neither 4 nor 5 can share a cell with 9, and must therefore share a cell with 1, leading us to the same contradiction.

Finally we consider the case if $\{1, 2, 6, 9, 10\}$ were a section for a partition. Neither 3 nor 5 can share a cell with 2 or 6 because they are in the same orbit under H_1. Likewise neither 3 nor 5 can share a cell with 9 since they are in the same orbit under H_2. Lastly neither 3 nor 5 can share a cell with 10, because $10^{H_6} = \{1, 3, 5, 7, 8, 10\}$. So 3 and 5 would have to share a cell with 1, for a contradiction of uniformity once more.

Therefore since all possible sections of size 2 or 5 have been accounted for up to their orbits under H, we can conclude that it is impossible to have a section-regular partition for H, and therefore H is synchronizing. $\square$

3.3 Basic Groups

The property of synchronization is also closely tied to the preservation of Cartesian structures via the concept of *basic groups*. To understand this relationship, we will start by defining a Cartesian structure [1].

Definition 3.11. A *Cartesian Structure* on Ω is a bijection between Ω and the set K^M of functions from M to K where $|M|, |K| > 1$.

If we take $M = \{1, \ldots, m\}$ and $K = \{1, \ldots, k\}$, then the automorphism group of a Cartesian structure as defined above is the wreath product $S_k \wr S_m$ with the product action.

Definition 3.12. Let $G \leq S_\Omega$. We say that G is *non-basic* if it preserves a Cartesian structure on Ω and *basic* otherwise.

We can thus make a more generalized statement about the structure of a non-basic group.

Corollary 3.13. *A non-basic group as defined above is embedded in the wreath product $S_k \wr S_m$.*

Once again, we have a property defined by *not* preserving some structure. Primitive groups don't preserve block systems, synchronizing groups don't preserve sections amongst partitions, and basic groups don't preserve a Cartesian structure. It turns out that synchronizing groups also do not preserve Cartesian structures. This is easiest to see with the monoid definition of synchronizing, that G is synchronizing if $\langle G, f \rangle$ has an element of rank 1 (meaning $|\text{im}(f)| = 1$), for a non-permutation f.

Lemma 3.14. *A synchronizing group is basic* [1].

Proof. Proof Let G be non-basic, and suppose that Ω has been identified with the set of m-tuples over a set A of size k, in such a way that G preserves the identification (and so is embedded in $S_k \wr S_m$ by Corollary 3.13). Let f be the map which takes the m-tuple $(a_1, a_2, ..., a_m)$ to the m-tuple $(a_1, a_1, \ldots, a_1)$ with all entries equal. Let B be the image of f. Then applying any element of G to B gives a set of k elements whose projections onto any coordinate form the whole of A; so following this by f gives us the set B again. Then applying any combination of elements of G and f still returns B. So no element of the monoid $\langle G, f \rangle$ can have rank (image size) smaller than k, thus G is nonsynchronizing. $\square$

Chapter 4

Synchronization of Groups of Affine Type

Looking at the O'Nan–Scott Theorem (Theorem 2.34) we recall that one class of primitive groups is the affine type. That is to say, if G is a primitive group of affine type of degree n, then $\mathrm{Soc}(G)$ is an elementary abelian p-group with $n = p^m$, and G is isomorphic to a subgroup of the affine group $AGL_m(p)$ including the translations. Additionally, the stabilizer G_α acts faithfully and irreducibly on the socle, interpreted as the natural module of $GL_m(p)$ [5]. Thus, this group has the form $G = \langle H, T\rangle$ where H is isomorphic to a matrix group $\tilde{H} \leq GL_m(\mathbb{F}_p)$ and T corresponds to the vector translations of $\mathbb{F}_p^m$. The permutation action is thus the action on the vectors in the vector space. We let each matrix in $\tilde{H}$ correspond to the permutation in H determined by where $\tilde{H}$ sends the vectors of $\mathbb{F}_p^m$. Additionally, T is the set of all permutations of translations in the vector space (where $t_w(v) = v + w$). Without loss of generality, we can choose a translation t corresponding to adding a 1 in the first position, and since H acts irreducibly, $G = \langle H, T\rangle = \langle h_1, \ldots, h_l, t\rangle$ for $\{h_1, \ldots, h_l\}$ the generators of H. We will denote H as the *matrix part* of G.

Example 4.1. Define $G = \langle H, T\rangle$ with matrix part

$$\langle h\rangle = H \cong \tilde{H} = \langle M\rangle = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \leq GL_2(\mathbb{F}_2).$$

We will assign a lexicographic labelling to the elements of $\mathbb{F}_2^2$ as follows:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto 1; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto 2; \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto 3; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto 4$$

Then based on where M sends these vectors, we get that $M \mapsto (2, 3, 4)$. Additionally we will consider the translation t that adds 1 to the first entry of each vector, resulting in $t \mapsto (1, 3)(2, 4)$. Thus $G = \langle (2, 3, 4), (1, 3)(2, 4)\rangle$ with matrix part $H = \langle (1, 3)(2, 4)\rangle$.

The structure of these groups having matrix parts gives us the powerful tools of linear algebra to use in our analysis of the Affine type primitive groups. We will make specific use of the linear action of the stabilizer to show properties of the permutation groups they correspond to.

Our next question is which groups could have such a form? It turns out that a theorem by Aschbacher describes such groups in a manner similar to the O'Nan–Scott Theorem, namely Aschbacher's Theorem (1984) [7]. The theorem states that subgroups of $GL_n(q)$ must fall into one of nine classes. We will use the "rough descriptions" of eight of these classes as given by Bray, Holt and Roney-Dougal in 2013 in Table 4.1 [8]. Not included in their table is the ninth class, which we will call $\mathcal{C}_9$, which is concerned with almost simple groups.

Table 4.1: Rough descriptions of Aschbacher classes from [8].

$\mathcal{C}_i$	Rough description
$\mathcal{C}_1$	stabilizers of totally singular or non-singular subspaces
$\mathcal{C}_2$	stabilizers of decompositions $V = \bigoplus_{i=1}^{t} V_i$, $\dim(V_i) = n/t$
$\mathcal{C}_3$	stabilizers of extension fields of $\mathbb{F}_{q^u}$ of prime index dividing n
$\mathcal{C}_4$	stabilizers of tensor product decompositions $V = V_1 \otimes V_2$
$\mathcal{C}_5$	stabilizers of subfields of $\mathbb{F}_{q^u}$ of prime index
$\mathcal{C}_6$	normalizers of symplectic-type or extraspecial groups in absolutely irreducible reps.
$\mathcal{C}_7$	stabilizers of decompositions $V = \bigotimes_{i=1}^{t} V_i$, $\dim(V_i) = 1, n = a^t$
$\mathcal{C}_8$	groups of similarities of non-degenerate classical forms

Thus, on our way to determining which primitive affine groups are synchronizing, we will distinguish between the Aschbacher classes for the stabilizer of the group. In the upcoming sections, we will determine what can be said about the synchronization of each of these classes. However before considering subgroups of $AGL_m(p)$, we will consider $AGL_m(p)$ itself as the parent of the group for the first part of the O'Nan–Scott Theorem. To do this, we recall that 2-transitivity implies synchronization. Thus in the context of the affine type, we can give the following statement indicating these groups as 2-transitive.

Lemma 4.2. *Let $G = \langle H, T\rangle$ where the matrix part H is isomorphic to a subgroup of $\tilde{H} \leq GL_m(\mathbb{F}_p)$. If $\tilde{H}$ acts transitively on the nonzero vectors in $\mathbb{F}_p^m$, then G is a 2-transitive permutation group.*

Proof. Note that by construction, $H = G_0$, the point stabilizer of the zero vector. Then take four vectors $x \neq y$ and $v \neq w$ in Ω. We want to show there exists some g such that $(x, y)^g = (x^g, y^g) = (v, w)$. Consider the translations t and s where t translates by $-x$ and s translates by $+v$, as well as $k \in G_0$ such that $(y^t)^k = w^{s^{-1}}$. Let $g = tks$. Without loss of generality, let $y \neq 0$. Then

$$\begin{aligned}
(x, y)^g &= (x, y)^{tks} \\
&= (x^t, y^t)^{ks} \\
&= (0, y^t)^{ks} && \text{(by assumption on } t) \\
&= (0^k, y^{tk})^s && \text{(note } y^t \neq 0 \text{ since } y \neq x) \\
&= (0, y^{tk})^s && \text{(since } k \in G_0) \\
&= (0^s, y^{tks}) \\
&= (v, w) && \text{(by assumptions on } k \text{ and } s)
\end{aligned}$$

To address the supposition that $y \neq 0$, we could consider $(y, x)^g = (w, v)$. Thus, this group is 2-transitive on the vector space. □

This theorem will help us in a few specific cases that we see later, but more immediately we can use this for the full affine case, since $SL_m(p)$ is transitive on nonzero vectors.

Corollary 4.3. *The actions of $ASL_m(p)$ and $AGL_m(p)$ on $\mathbb{F}_p^m$ are 2-transitive.*

Furthermore, 2-transitivity of a group implies synchronization.

Corollary 4.4. *The actions of $ASL_m(p)$ and $AGL_m(p)$ on $\mathbb{F}_p^m$ are synchronizing.*

We now will consider the synchronization of the subgroups of $AGL_m(p)$.

4.1 Class $\mathcal{C}_1$: The Reducible Case

In this class, the matrix part acts reducibly, and therefore the corresponding permutation groups are imprimitive per the O'Nan–Scott classification. These are thus always nonsynchronizing.

4.2 Class $\mathcal{C}_2$: The Wreath Product Case

Our first new result comes from class $\mathcal{C}_2$. This gives us a matrix part of the group which is formed by a wreath product. Here we can generate the matrix part with block diagonal matrices with n blocks of size k, and permutation matrices that permute the blocks. Then for the overall action on the vector space, we can break down the action into translations, permutations of length k pieces of vectors, and block diagonal actions of matrices on the vectors of length nk. To get a better understanding of how the matrix part $GL_k(p) \wr S_n$ behaves, we will look at an example.

Example 4.5. Take $GL_2(5) \wr S_3$, a group of 6×6 matrices over $\mathbb{F}_5$. Then $GL_2(5) \wr S_3$, is naturally split into two parts, the base and the wreath. First, consider the following two block matrices g_1, g_2.

$$g_1 = \left(\begin{array}{cc|cc|cc} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad g_2 = \left(\begin{array}{cc|cc|cc} 4 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Note that the upper blocks of these matrices are generators of $GL_2(5)$. Thus together these matrices generate a piece of the base group

$$\langle g_1, g_2 \rangle \cong GL_2(5) \times E \times E$$

where E is the trivial subgroup of $GL_2(5)$. To get the rest of the group, we introduce the block permutation matrices, which together generate a group isomorphic to S_3. Consider the block matrices h_1, h_2.

$$h_1 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad h_2 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

These matrices permute the entries of a vector or matrix, keeping each block of $k = 2$ together. Thus by themselves,

$$\langle h_1, h_2 \rangle \cong \langle (1,2), (1,2,3) \rangle \cong S_3$$

they can permute each GL factor. Then all together, we get

$$\langle g_1, g_2, h_1, h_2 \rangle \cong GL_2(5) \times GL_2(5) \times GL_2(5) \rtimes S_3 = GL_2(5) \wr S_3$$

i.e. the entire wreath product. A natural way to view these actions is by dividing our vectors up into 3 *subvectors* of length 2. More generally, n subvectors of length k.

Using this view, we are able to achieve new results in showing that primitive groups where the matrix part has this form are never synchronizing.

Theorem 4.6. *If G is of affine type with matrix part $GL_k(p) \wr S_n$, then G is nonsynchronizing.*

Proof. Note that G acts on the vector space $\mathbb{F}_p^{nk}$. We will use the notion of subvectors to describe these $n \cdot k$ length vectors as the concatenation of n subvectors of length k. Consider section of vectors,

$$T := \left\{ \begin{bmatrix} v_1 \\ \hline v_1 \\ \hline \vdots \\ \hline v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \hline v_2 \\ \hline \vdots \\ \hline v_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_{p^k} \\ \hline v_{p^k} \\ \hline \vdots \\ \hline v_{p^k} \end{bmatrix} \right\}$$

where every vector is n copies of the same subvector, and the partition P given by the equivalence relation $u \sim w$ if and only if u and w are equal in the first subvector.

Now, the elements of G are composed of three types of actions on the vectors in $\mathbb{F}_p^{n \cdot k}$ – the permutations of subvectors given by S_n, the action of the block diagonal matrices, and the permutations given by vector addition. We want to verify that T^g is a section for P for all g in G. Note that for any g in G, we can write g as $g = r \cdot d \cdot a$ for r, d, a corresponding to some row permutation, diagonal multiplication, and addition of some vector respectively. Then $T^g = T^{rda} = ((T^r)^d)^a$. First we note that $T^r = T$ as, a permutation of equal subvectors, and therefore T^r is a section for P. Second, consider $(T^r)^d = T^d$, the image of T under block diagonals. Note that each block is a matrix in $GL_n(p)$ and each block acts on the subvectors independently. Namely, the matrix in the first block will act on the first subvector of each vector in T. Then since an element of $GL_k(p)$ acting on the elements of $\mathbb{F}_p^k$ will permute them, we get that the vectors in T^d are all distinct in the first subvector, and therefore a section for P. Third, we consider $T^g = (T^d)^a$, the image of T^d we get by adding some vector v to each vector. Once again we will focus on the first subvector of v and the vectors in T. Then the images of the first subvectors of elements in T will be permuted by the addition of v, and therefore T^{rda} is a section for P. Thus, G is nonsynchronizing. $\square$

Furthermore, by the contrapositive of Lemma 3.7 any subgroup of such a group must also be nonsynchronizing, with the same section and partition.

4.3 Class $\mathcal{C}_3$: The Extension Field Case

We did not yet explore this class beyond initial concrete examples, the smallest of which were synchronizing. The nature of these groups means that we are not able to say they are nonsynchronizing just because of the structure.

4.4 Class $\mathcal{C}_4$: The Kronecker Product Case

Our second new result comes from class $\mathcal{C}_4$. This gives us a matrix part of the group which is formed by a Kronecker Product of matrices. Then in addition to the vector translations, our group has actions from matrices $GL_m(p) \otimes GL_k(p)$ acting on vectors in $\mathbb{F}_P^{mk}$ with the property of $(A \otimes B) \cdot (v \otimes w) = (A \cdot v) \otimes (b \cdot w)$.

Theorem 4.7. *If G is a primitive group of affine type with the matrix part of G being the Kronecker product $GL_m(p) \otimes GL_k(p)$, then G is nonsynchronizing.*

Proof. Consider $GL_m(p) \otimes GL_k(p)$ acting on the space $V \otimes W$. Without loss of generality, we will choose $m \leq k$ or equivalently, $|V| \leq |W|$. We will show that for a fixed $v_0 \in V$, (we'll choose $v_0 = [0, 0, \ldots, 0, 0, 1]$) there exists a set S and partition $\mathcal{P}$ such that the latter is section-regular under G with section S. To do so, we will define the subspace $S := \{v_0 \otimes w \mid w \in W\}$ corresponds to a section for G. To define $\mathcal{P}$, we need a complementary subspace P_0 such that $\langle P_0, S \rangle = V \otimes W$ and $P_0 \cap S = \langle \, \rangle$, and define the partition $\mathcal{P} := \{P[z] \mid z \in S\}$ with $P[z] := \{t + z \mid t \in P_0\}$. It follows from the structure that P_0 and S are complementary, and S is a section for $\mathcal{P}$. Note that S is the set of vectors with the first $k(m-1)$ entries 0. To get $\mathcal{P}$, we define P_0 as the subspace with the rows of the following block matrix as a basis.

$$\left(\begin{array}{c|c|c|c|c|c}
I_k & 0 & 0 & \cdots & 0 & M_1 \\
\hline
0 & I_k & 0 & \cdots & 0 & M_2 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & \cdots & 0 & I_k & M_{m-1}
\end{array}\right)$$

We will define $\mathcal{M} = \{I_k, M_1, M_2, \ldots, M_{m-1}\}$ to be a subset of a k-dimensional matrix representation of $\mathbb{F}_{p^m}$ (since $|\mathcal{M}| = m \leq k$). By Theorem 4.8 below, each nonzero vector in the space P_0 is nondecomposable. In other words, $P_0 \cap \{v \otimes w \mid v \in V, w \in W\} = \langle\, \rangle$.

With S the section described, we want to show S^g is a section for any $g \in G$. Define the action of g on $v \in V \otimes W$ by $v^g := Mv + x$ for some $M = A \otimes B \in GL_m(p) \otimes GL_k(p)$ with a translation by $x \in V \otimes W$. Let $s, r \in S$ with the forms $v_0 \otimes w_s$ and $v_0 \otimes w_r$. Now suppose that s^g and r^g are both elements of the cell $P[z]$. This implies that $s^g - z$ and $r^g - z$ are elements of the subspace $P_0 = P\left[\vec{0}\right]$. Thus we will look at the difference of these vectors which must also be in P_0 by closure.

$$\begin{aligned}
(s^g - z) - (r^g - z) &= (Ms + x - z) - (Mr + x - z) \\
&= Ms - Mr \\
&= M(s - r) \\
&= (A \otimes B)(v_0 \otimes w_s - v_0 \otimes w_r) \\
&= (A \otimes B)(v_0 \otimes (w_s - w_r)) \\
&= Av_0 \otimes B(w_s - w_r)
\end{aligned}$$

We observe that $Av_0 \otimes B(w_s - w_r)$ is clearly a decomposable tensor in P_0, but by construction, the only decomposable tensor in P_0 is the zero vector. Since A is invertible and v_0 is nonzero, we must have $B(w_s - w_r) = \vec{0}$. And thus, since B is also invertible, we get $w_s = w_r$, and thus $s = r$. Therefore S^g must intersect at least $|S|$ different cells, and by the pigeonhole principle S^g is a section. Thus G is nonsynchronizing. $\square$

Once again, by the contrapositive of Lemma 3.7 any subgroup of such a group must also be nonsynchronizing, with the same section and partition. However, to complete the proof, we need the following theorem to guarantee the existence of a subspace that does not contain any nondecomposable tensors.

Theorem 4.8. *Let $\mathcal{M} = \{I_k, M_1, M_2, \ldots, M_{m-1}\}$ such that $M_i \in GL_k(p)$ and take rows of B as defined below to form a basis for a $k(m-1)$ dimensional space P. If rank$(A) = k$ for any linear combination of matrices A in $\mathcal{M}$, then P does not contain any nonzero decomposable tensors of the form $v \otimes w \in \mathbb{F}_p^m \otimes \mathbb{F}_p^k$ with $m \leq k$.*

$$B = \left(\begin{array}{c|c|c|c|c|c} I_k & 0 & 0 & \cdots & 0 & M_1 \\ \hline 0 & I_k & 0 & \cdots & 0 & M_2 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & I_k & M_{m-1} \end{array} \right)$$

Proof. In search of a contradiction, given the above matrix with said properties to be the rows of our basis, suppose there exists a tensor product $x = \hat{\alpha} \otimes \hat{\delta}$ in our space. We will take $\hat{\delta} = [\delta_1, \ldots, \delta_k]$ and $\hat{\alpha} = [\alpha_1, \ldots, \alpha_i, \ldots \alpha_m]$. Let us denote the basis vectors as

$$\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_{2k}, \ldots, b_{(m-2)k+1}, \ldots, b_{(m-1)k}\}.$$

The construction of the basis gives us unique vectors determined by the first $k(m-1)$ entries. Therefore if x is decomposable, $x = \hat{\alpha} \otimes \hat{\delta}$, then $\hat{\delta}$ is determined by the first $k(m-1)$ entries. That is to say if entries $(i-1)k+1$ through ik of x are $\alpha_i \cdot \hat{\delta}$, then $\alpha_i \cdot \hat{\delta}$ comprises the $(i-1)k+1$ through ik entries of the coefficient vector for x. Therefore, if the last k entries of x are $\alpha_m \hat{\delta}$, then we can describe these entries as follows:

$$\alpha_m \cdot \hat{\delta} = \sum_{i=1}^{m-1} \alpha_i M_i^T \cdot \hat{\delta}$$

Then the following must hold

$$
\begin{aligned}
0 &= -\alpha_m I \cdot \hat{\delta} + \left(\sum_{i=1}^{m-1} \alpha_i M_i^T \cdot \hat{\delta} \right) \\
&= -\alpha_m I \cdot \hat{\delta} + \left(\sum_{i=1}^{m-1} \alpha_i M_i^T \right) \cdot \hat{\delta} \\
&= \left(-\alpha_m I + \sum_{i=1}^{m-1} \alpha_i M_i^T \right) \cdot \hat{\delta} \\
&= \left(-\alpha_m I + \sum_{i=1}^{m-1} \alpha_i M_i \right)^T \cdot \hat{\delta}
\end{aligned}
$$

Which would imply that the matrix $-\alpha_m I + \sum_{i=1}^{m-1} \alpha_i M_i$ (which is a linear combination in $\mathcal{M}$) does not have full rank, giving us a contradiction. □

4.5 Class $\mathcal{C}_5$: The Non-prime Field Case

We recall the O'Nan–Scott Theorem to see that the primitive cases only occur for matrix actions over prime fields. Therefore this class is not applicable as there will be no primitive groups of affine types with a matrix part of this form.

4.6 Class $\mathcal{C}_6$: The Normalizers Case

We have not looked at this class due to time constraints. One issue is that the tools necessary here will differ greatly from what we been used in $\mathcal{C}_2$ and $\mathcal{C}_4$ due to the difference in structure.

4.7 Class $\mathcal{C}_7$: The Tensor Wreath Case

Our third new result comes from class $\mathcal{C}_7$. This gives us a matrix part of the group which is formed by a tensor product of matrices. Then in addition to the vector translations, our group has actions from matrices $GL_m(p) \wr S_n$ acting on vectors in $\mathbb{F}_p^{mn}$. We will use the same proof as Theorem 4.7 with a minor change to include the S_n part of the wreath product. We can do this by noticing that n tensored copies of $GL_m(p)$ can be written as $GL_m(p) \otimes GL_n(p)$ (as in Theorem

4.7) where $k = m(n-1)$. The key addition to this proof lies in the final equality where we must multiply by a matrix H indicating the permutation of blocks.

Theorem 4.9. *If G is a primitive group of affine type with the matrix part of G being the tensor wreath product $GL_m(p) \otimes GL_m(p) \otimes \cdots \otimes GL_m(p) \rtimes S_n$ then G is nonsynchronizing.*

Proof. The first thing we will do is rewrite $GL_m(p) \otimes GL_m(p) \otimes \cdots \otimes GL_m(p)$ as $GL_m(p) \otimes GL_k(p)$ with $k = m(n-1)$ which acts on the space $V \otimes W$. We will show that for a fixed $v_0 \in V$, (we'll choose $v_0 = [0, 0, \ldots, 0, 0, 1]$) there exists a set S and partition $\mathcal{P}$ such that the latter is section-regular under G with section S. To do so, we will define the subspace $S := \{v_0 \otimes w \mid w \in W\}$ corresponds to a section for G. To define $\mathcal{P}$, we need a complementary subspace P_0 such that $\langle P_0, S \rangle = V \otimes W$ and $P_0 \cap S = \langle\ \rangle$, and define the partition $\mathcal{P} := \{P[z] \mid z \in S\}$ with $P[z] := \{t + z \mid t \in P_0\}$. It follows from the structure that P_0 and S are complementary, and S is a section for $\mathcal{P}$. Note that S is the set of vectors with the first $k(m-1)$ entries 0. To get $\mathcal{P}$, we define P_0 as the subspace with the rows of the following block matrix as a basis.

$$\left(\begin{array}{c|c|c|c|c|c}
I_k & 0 & 0 & \cdots & 0 & M_1 \\ \hline
0 & I_k & 0 & \cdots & 0 & M_2 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \cdots & 0 & I_k & M_{m-1}
\end{array}\right)$$

We will define $\mathcal{M} = \{I_k, M_1, M_2, \ldots, M_{m-1}\}$ to be a subset of a k-dimensional matrix representation of $\mathbb{F}_{p^m}$ (since $|\mathcal{M}| = m \leq k$). By Theorem 4.8, each nonzero vector in the space P_0 is nondecomposable. In other words, $P_0 \cap \{v \otimes w \mid v \in V, w \in W\} = \langle\ \rangle$.

With the section S described, we want to show S^g is a section for any $g \in G$. Define the action of g on $v \in V \otimes W$ by $v^g := Mv + x$ for some $M = H(A \otimes B) \in GL_m(p) \otimes GL_k(p) \rtimes S_n$ with a translation by $x \in V \otimes W$, where H is the matrix from S_n permuting blocks. Let $s, r \in S$ with the forms $v_0 \otimes w_s$ and $v_0 \otimes w_r$. Now suppose that s^g and r^g are both elements of the cell $P[z]$. This implies that $s^g - z$ and $r^g - z$ are elements of the subspace $P_0 = P\left[\vec{0}\right]$. Thus we will look at the difference of these vectors which must also be in P_0 by closure.

$$
\begin{aligned}
(s^g - z) - (r^g - z) &= (Ms + x - z) - (Mr + x - z) \\
&= Ms - Mr \\
&= M(s - r) \\
&= H(A \otimes B)(v_0 \otimes w_s - v_0 \otimes w_r) \\
&= H(A \otimes B)(v_0 \otimes (w_s - w_r)) \\
&= H(Av_0 \otimes B(w_s - w_r)) \\
&= H(Av_0 \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_n) && \text{(identified back to } n \text{ blocks)} \\
&= H(x_1 \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_n) \\
&= x_{1^h} \otimes x_{2^h} \otimes \cdots \otimes x_{n^h} && (h \text{ the permutation from } H) \\
&= x_{1^h} \otimes (x)
\end{aligned}
$$

We observe that $x_{1^h} \otimes (x)$ is clearly a decomposable tensor in P_0, but by construction, the only decomposable tensor in P_0 is the zero vector. Since A is invertible and v_0 is nonzero, then we must have $B(w_s - w_r) = \vec{0}$. As in the proof of Theorem 4.7, since B is also invertible, we get $w_s = w_r$, and thus $s = r$. Therefore S^g must intersect at least $|S|$ different cells, and by the pigeonhole principle S^g is a section. Thus G is nonsynchronizing. □

Similarly to before, we use the contrapositive of Lemma 3.7 to say any subgroup of such a group must also be nonsynchronizing, with the same section and partition.

4.8 Class $\mathcal{C}_8$: Non-degenerate Forms Case

From class $\mathcal{C}_8$ we have non-degenerate classical forms. This itself will spawn a series of subclasses. For one of the classes (symplectic forms) we will show in the following section that these groups are always synchronizing. The remaining forms require further consideration and case distinction. At minimum, our computational evidence has shown that some groups from orthogonal forms are synchronizing, while others are nonsynchronizing. We expect that the requirements here will depend on the degree of the group - both in parity and magnitude.

4.8.1 Symplectic Forms

We argue that the full affine symplectic groups are 2-transitive and therefore synchronizing. Specifically, we want to employ Lemma 4.2, and thus need to show that the groups from symplectic forms sufficiently move the vectors.

From Hulpke [9] due to work by Birman [10] and Klingen [11], we have that the symplectic group has a generating set given by

$$\mathrm{Sp}_{2n}(\mathbb{Z}_p) = \langle Y_i, U_i, Z_j \mid 1 \leq i \leq n, 1 \leq j \leq n-1 \rangle,$$

where $Y_i = t_{i,n+i}^{-1}$, $U_i = t_{n+i,i}$ and $Z_i = \begin{bmatrix} I_n & B_i \\ 0 & I_n \end{bmatrix}$ for B_i the matrix with $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ at $i, i+1$ along the diagonal, and all other entries zero. Here $t_{i,j}$ is defined for $i \neq j$ to be the matrix that has ones along the diagonal and at position (i,j), and zeroes elsewhere.

We also note that for $1 \leq i \neq j \leq n$ the products $(t_{i,j})(t_{n+j,n+i}^{-1})$ form a subgroup P where

$$P = \left\{ \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix} \mid M \in \mathrm{SL}_n(\mathbb{Z}_p) \right\}.$$

Lemma 4.10. *The symplectic group $Sp_{2n}(\mathbb{Z}_p)$ as defined above acts transitively on the nonzero vectors in $\mathbb{F}_p^{2n}$.*

Proof. In order to show that $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ is transitive, it will suffice to show that any nonzero vector w can be sent to $v_1 = (1, 0, 0, \ldots, 0)$. We will consider two cases for w

- If the first n entries of w are zero, then there is some element $p \in P$ such that w^p is the vector with zeroes everywhere except for a 1 in the $n+1$ entry. Then for $y = Y_1^{-1} = t_{1,n+1}$ we have w^{py} is the vector with zeroes everywhere except for the ones in the 1 and $n+1$ entries. Then for $u = U_1^{-1}$ we get $w^{pyu} = v_0$.

- If there are any nonzero entries in the first n positions of w, there exists a $p \in P$ such that w^p has all ones in the first n positions. Then for each i with a nonzero element in the $n+i$ position, we can apply U_i as many times as needed until that position goes to zero. Consider $u = \prod_{w^p_{n+i} \neq 0} (U_i)^{a_i}$. Then w^{pu} is the vector with the first n entries 1, and remaining entries 0. Then there exists some $q \in P$ such that $w^{puq} = v_1$.

We have shown that for any w, there exists a group element $g_w \in \mathrm{Sp}_{2n}(\mathbb{Z}_p)$ such that $w^g = v_1$. Therefore for any two nonzero vectors w, z we have $w^{g_w g_z^{-1}} = z$, and therefore $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ is transitive on the nonzero vectors of $\mathbb{F}_p^{2n}$.

□

The above with Lemma 4.2 lets us determine that the group G consisting of the group generated by $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ with all of the vector translations is 2-transitive on all of the vectors in $\mathbb{F}_p^{2n}$ (not just the nonzero ones).

Corollary 4.11. *If G is a primitive group of affine type with the matrix part of G being $Sp_{2n}(\mathbb{Z}_p)$, then G is 2-transitive on $\mathbb{F}_p^{2n}$.*

We can thus extend this to synchronizing.

Corollary 4.12. *If G is a primitive group of affine type with the matrix part of G being $Sp_{2n}(\mathbb{Z}_p)$, then G is synchronizing.*

4.8.2 Orthogonal Forms

We did not study these analytically, but have some preliminary results from computation. These results are given in Table 4.2 for prime fields with $p > 2$ (due to the nature of the orthogonal groups, the $p = 2$ case should be handled independently).

Table 4.2: Example data from GAP for orthogonal forms.

Matrix part	Degree	GAP ID	Synchronizing?
GO$(0,3,3)$	8	27	No
GO$(0,3,5)$	125	42	No
GO$(0,5,3)$	243	28	No
GO$(-1,2,3)$	9	2	No
GO$(-1,2,5)$	25	7	No
GO$(-1,4,3)$	81	138	Yes
GO$(-1,4,5)$	625	656	Yes
GO$(-1,6,3)$	729	481	Yes
GO$(+1,2,3)$	9	N/A	No
GO$(+1,2,5)$	25	5	No
GO$(+1,4,3)$	81	103	No
GO$(+1,4,5)$	625	651	No
GO$(+1,6,3)$	729	478	No

Due solely from the data in Table 4.2, we ask the following open questions:

- If G is of affine type with matrix part GO$(0, 2n+1, p)$, with $p > 2$, is G always nonsynchronizing?
- If G is of affine type with matrix part GO$(-1, 2n, p)$ with $p > 2$ and $n > 1$, is G always synchronizing?
- If G is of affine type with matrix part GO$(+1, 2n, p)$ with $p > 2$ and $n > 1$, is G always nonsynchronizing?

4.9 Class $\mathcal{C}_9$: The Almost Simple Case

Recall Definition 2.32, that a group is *almost simple* if its socle is simple. Since the simple groups themselves do not share an underlying structure, it makes giving a structure based statement about the entire class impossible. With a separate case distinction for each of the types of simple groups (as stabilizers of affine groups), one might be able to make progress into determining synchronization for these groups. However, we have not looked at these as a whole (past the ones that overlap different classes).

Chapter 5

Beyond the Affine Type

Determining synchronization of primitive groups extends beyond just the groups of affine type. As affine type was the first case from the O'Nan–Scott Theorem, let us recall the non-affine types from Theorem 2.34:

The socle H is isomorphic to a direct product T^m of a nonabelian simple group T with one of the following:

1. $m = 1$ and G is isomorphic to a subgroup of Aut(T).
2. $m \geq 2$ and G is a subgroup of diagonal type with $n = |T|^{m-1}$.
3. $m \geq 2$ and there exists some proper divisor d of m and some primitive group U such that Soc(U)$\cong T^d$, and G is isomorphic to a subgroup of $U \wr S_{m/d}$ with the product action, and $n = l^{m/d}$ where l is the degree of U.
4. $m \geq 6$ with H regular and $n = |T|^m$.

This gives us four additional cases to consider, where we don't have the benefit of a geometric relation. In these cases, Peter Cameron and his colleagues have solved the synchronizing problem for a few types. One helpful tool to understand for this is the property of separating groups.

Definition 5.1. We say that G is *non-separating* if given nontrivial subsets A, B of Ω satisfying $|A| \cdot |B| = |\Omega|$ for all g in G, we have $|A^g \cap B| = 1$. We say that G is *separating* otherwise [1].

The similarities between this definition and that of synchronizing come quickly, with A assuming the role of the transversal, and B a cell in the partition. Moreover, all separating groups are synchronizing [1]. The few counterexamples prove to be difficult. The ultimate benefit of separation comes from the following theorem:

Theorem 5.2. *Let G be a primitive group which is not almost simple. Then G is synchronizing if and only if it is separating [12].*

Therefor, for all O'Nan–Scott Types barring the almost simple groups, we can use the stronger property of separating where applicable (though we didn't need this for the affine type, since we were using a geometric approach).

5.1 Class 1: Almost Simple Type

The case that G is isomorphic to a subgroup of the automorphism group of its socle gives us the almost simple groups. The generic tools that we have been using (and tools that have been used for the other classes such as the separation property) cannot be applied in this scenario. Due to the nature of the almost simple groups (as inherited from the finite simple groups), a universal conclusion covering the entire class is impossible. This means that the analysis of synchronization for the almost simple groups will be dependent on the type of the group, and not solely the fact that it is almost simple. With some luck, this could come down to analyzing the simple groups socles by class (of which there are 18 plus the 26 sporadic groups). This could result in a case distinction of over 44 cases (if not more) which lies beyond our scope here.

5.2 Class 2: Diagonal Type

For groups of diagonal type, we can paint a clearer picture. This is largely due to the work of Bray, Cai, Cameron, Spiga, and Zhang leveraging the separation property and the proof of the Hall-Paige Conjecture [12]. Their result is as follows:

Theorem 5.3. *Let G be a primitive permutation group of simple diagonal type with more than two factors in the socle. Then G is nonsynchronizing.*

Conveniently, this gives us an O'Nan–Scott Type which is entirely non-synchronizing – our first complete puzzle piece that doesn't require several case distinctions.

5.3 Class 3: Wreaths with the Product Action

Wreath products with the product action raise another case of nonsynchronizing. This result is alluded to in [1], and here we can construct the section-regular partition without need for the separation property.

Theorem 5.4. *A primitive group with product action is nonsynchronizing.*

Proof. Let G act primitively with the product action on Ω. Then we can identify Ω as a set of m-tuples over a set A of size k, that is to say $G \leq S_k \wr S_m$. Let P be the partition on Ω inherited by the equivalence relation of $x \cong y$ if x and y are equal in the first component. Consider the set of m-tuples $S := \{(a, a, \ldots, a) \mid a \in A\}$. Then clearly S is a section for P with $|S| = k$ such that P has k cells. Then take S^g for any $g \in G$. Note that since each element of S differs in the first element, their image under g will all be different in the first element, therefore being in their own cell in P. Therefore S^g is a section for P. Then P is a section-regular partition for G with section S and thus G is nonsynchronizing. $\square$

This is a construction that is a bit simpler to imagine than our results from the affine cases. This third class gives us yet another complete piece of the puzzle: an O'Nan–Scott Type which is always nonsynchronizing.

5.4 Class 4: Twisted Wreath Products

Alas, we arrive at the twisted wreath products. We note that by the O'Nan–Scott classification, these groups have a socle containing the product of at least 6 copies of a regular nonabelian simple group. Thus, the smallest group in this case will have degree of $60^6 = 46{,}656{,}000{,}000$. On one hand, these permutation groups are too large to be able to experiment with. On the other hand – their size also makes them quite impractical to study for having section-regular partitions.

Chapter 6

Conclusion

Overall, the "synchronizing problem" is now closer to being solved. We have several cases of primitive groups that can be classified as synchronizing or nonsynchronizing as determined by their group structure. To review the cases of affine type, recall that we divided amongst the Aschbacher Classes with the following conclusions.

- $\mathcal{C}_1$: This class is irreducible, and therefore imprimitive per the O'Nan–Scott classification. Thus, these groups are **always nonsynchronizing**.

- $\mathcal{C}_2$: This is class gives our first new result, Theorem 4.6. Groups coming from this class are **always nonsynchronizing**. We have additionally provided a construction for the section-regular partitions that show this.

- $\mathcal{C}_3$: We did not have time to explore this case. Our elementary experimental data has shown that the groups from this class are **sometimes synchronizing**, and will thus require further analysis.

- $\mathcal{C}_4$: This class gives our second new result, Theorem 4.7. Groups coming from this class are **always nonsynchronizing**. We have provided a construction for the section-regular partitions that show this.

- $\mathcal{C}_5$: Similarly to $\mathcal{C}_1$, we look to the O'Nan–Scott Theorem to see that primitive groups of affine type only occur over prime fields. Therefore groups of this form are imprimitive and thus **always nonsynchronizing**.

- $\mathcal{C}_6$: We did not have time to explore this case. The structure of these groups varies from the groups where we got results, and this will require some different tools. The synchronization of these groups is **unknown** and will require further analysis.

- $\mathcal{C}_7$: This is another class where were able to get a complete result. Using a combination of the techniques from cases $\mathcal{C}_2$ and $\mathcal{C}_4$, we were able to show that these groups are **always nonsynchronizing**. Once again, we provided a construction for the section-regular partitions to show this.

- $\mathcal{C}_8$: This is the final case where we have new results. Unfortunately, since this is the class of "stabilizers of forms," it requires further case distinctions. The proper result here is that the primitive groups of affine type coming from stabilizers of symplectic forms are **always synchronizing**. The remaining groups in this class - those stabilizing other kinds of forms - are **sometimes synchronizing**.

- $\mathcal{C}_9$: This case is the affine instance of almost simple. The synchronization of these groups will have to be determined on an individual basis, since the groups do not have the same underlying structure we were able to use in some of the other classes.

Thus, for primitive groups of affine type, we have five of the nine Aschbacher classes classified as synchronizing or nonsynchronizing, with partial progress on a sixth class that requires further case distinction. Moreover, for the classes determined to be nonsynchronizing, we have provided the specific construction of the sections and partitions that give us this property of the group. We had the advantage in doing this by the geometric relation of these groups, with the socles being identified to matrix groups acting on vector spaces.

Since the question of synchronization also regards the non-affine primitive groups as well, we will summarize the progress of those classes from the O'Nan–Scott Theorem (Theorem 2.34(2)):

- Almost Simple: Once again, due to the less coherent structure of the simple groups, analysis of the almost simple groups is difficult without individual case distinctions. The synchronization of these groups overall is **unknown**.

- Diagonal Type: Due to Bray et al. [12], using the property of separation and the proof of the Hall–Paige Conjecture, these groups have shown to be **always nonsynchronizing**.

- Wreath Product with Product Action: With a short proof (with construction of the section-regular partition in Theorem 5.4) we can show that the groups of this type are **always non-synchronizing**.
- Twisted Wreath Product: The smallest of these groups having degree 60^6 (having at least six copies of a regular nonabelian simple group as the socle), these groups are largely impractical to work with. The synchronization of groups in this class is **unknown**.

Thus, out of the five O'Nan–Scott types of primitive groups, we have two that are always nonsynchronizing, two that are largely unexplored, and one that has been divided into a further case distinction via Aschbacher classes - five of which have been determined. We have laid the groundwork for exploration of a few of the remaining Aschbacher classes. That combined with the number of combinatorialists working on this problem indicate that further progress is very promising in due course.

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